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Study of the Fractional Differential Problem of Some Matrix Fractional Functions

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Abstract: **In this paper, based on Jumarie type of Riemann-Liouville (R-L) fractional derivative and a new multiplication of fractional analytic functions, we can obtain arbitrary order fractional derivative of two types of matrix fractional functions by using some methods. In fact, our results are generalizations of ordinary calculus results.**

Keywords: **Jumarie type of R-L fractional derivative, new multiplication, fractional analytic functions, matrix fractional functions.**

I. INTRODUCTION

In the second half of the 20th century, a considerable number of studies on fractional calculus were published in the engineering literature. In fact, fractional calculus has many applications in physics, mechanics, biology, electrical engineering, viscoelasticity, control theory, economics, and other fields [1-17]. There is no doubt that fractional calculus has become an exciting new mathematical method to solve diverse problems in mathematics, science, and engineering.

Until now, the rules of fractional derivative are not unique. Many authors have given the definition of fractional derivative. The commonly used definition is the Riemann-Liouvellie (R-L) definition. Other useful definitions include Caputo definition of fractional derivative, Grunwald Letnikov (G-L) fractional derivative, conformable fractional derivative, and Jumarie's modified R-L fractional derivative [18-22]. Since Jumarie type of R-L fractional derivative helps to avoid non-zero fractional derivative of constant function, it is easier to use this definition to connect fractional calculus with traditional calculus.

In this paper, based on Jumarie type of R-L fractional derivative and a new multiplication of fractional analytic functions, we study the fractional differential problem of the following two types matrix fractional functions:

$$\begin{gathered} cos_\alpha(pAx^\alpha)\otimes_\alpha \tfrac{1}{2}Ln_\alpha(1+2rcos_\alpha(Ax^\alpha)+r^2) \\ -sin_\alpha(pAx^\alpha)\otimes_\alpha arctan_\alpha\left(rsin_\alpha(Ax^\alpha)\otimes_\alpha[1+rcos_\alpha(Ax^\alpha)]^{\otimes_\alpha(-1)}\right) \end{gathered},$$

and

$$\begin{gathered} cos_\alpha(pAx^\alpha)\otimes_\alpha arctan_\alpha\left(rsin_\alpha(Ax^\alpha)\otimes_\alpha[1+rcos_\alpha(Ax^\alpha)]^{\otimes_\alpha(-1)}\right) \\ +sin_\alpha(pAx^\alpha)\otimes_\alpha \tfrac{1}{2}Ln_\alpha(1+2rcos_\alpha(Ax^\alpha)+r^2) \end{gathered},$$

where $0<\alpha\le 1$, r is a real number, $|r|<1$, p is a positive integer, and A is a real matrix. Using some methods, we can evaluate arbitrary order fractional derivative of these two types of matrix fractional functions. Moreover, our results are generalizations of traditional calculus results.

II. PRELIMINARIES

At first, we introduce the fractional derivative used in this paper and its properties.

Definition 2.1 ([23])**:** Let $0<\alpha\leq 1$, and x_0 be a real number. The Jumarie type of Riemann-Liouville (R-L) α-fractional derivative is defined by

$$\left({}_{x_0}D_x^{\alpha}\right)[f(x)]=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{x_0}^{x}\frac{f(t)-f(x_0)}{(x-t)^{\alpha}}dt\ , \tag{1}$$

where $\Gamma(\ \ )$ is the gamma function. On the other hand, for any positive integer m, we define $\left({}_{x_0}D_x^{\alpha}\right)^m[f(x)]=\left({}_{x_0}D_x^{\alpha}\right)\left({}_{x_0}D_x^{\alpha}\right)\cdots\left({}_{x_0}D_x^{\alpha}\right)[f(x)]$, the m-th order α-fractional derivative of $f(x)$.

Proposition 2.2 ([24])**:** *If* α,β,x_0,C *are real numbers and* $\beta\geq\alpha>0$, *then*

$$\left({}_{x_0}D_x^{\alpha}\right)\left[(x-x_0)^{\beta}\right]=\frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(x-x_0)^{\beta-\alpha}, \tag{2}$$

and

$$\left({}_{x_0}D_x^{\alpha}\right)[C]=0. \tag{3}$$

Definition 2.3 ([25])**:** If x,x_0, and a_n are real numbers for all n, $x_0\in(a,b)$, and $0<\alpha\leq 1$. If the function $f_\alpha:[a,b]\to R$ can be expressed as an α-fractional power series, that is, $f_\alpha(x^\alpha)=\sum_{n=0}^{\infty}\frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}$ on some open interval containing x_0, then we say that $f_\alpha(x^\alpha)$ is α-fractional analytic at x_0. Furthermore, if $f_\alpha:[a,b]\to R$ is continuous on closed interval $[a,b]$ and it is α-fractional analytic at every point in open interval (a,b), then f_α is called an α-fractional analytic function on $[a,b]$.

In the following, we introduce a new multiplication of fractional analytic functions.

Definition 2.4 ([26])**:** If $0<\alpha\leq 1$. Assume that $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are two α-fractional power series at $x=x_0$,

$$f_\alpha(x^\alpha)=\sum_{n=0}^{\infty}\frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}, \tag{4}$$

$$g_\alpha(x^\alpha)=\sum_{n=0}^{\infty}\frac{b_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}. \tag{5}$$

Then

$$\begin{aligned}&f_\alpha(x^\alpha)\otimes_\alpha g_\alpha(x^\alpha)\\&=\sum_{n=0}^{\infty}\frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}\otimes_\alpha\sum_{n=0}^{\infty}\frac{b_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}\\&=\sum_{n=0}^{\infty}\frac{1}{\Gamma(n\alpha+1)}\left(\sum_{m=0}^{n}\binom{n}{m}a_{n-m}b_m\right)(x-x_0)^{n\alpha}.\end{aligned} \tag{6}$$

Equivalently,

$$\begin{aligned}&f_\alpha(x^\alpha)\otimes_\alpha g_\alpha(x^\alpha)\\&=\sum_{n=0}^{\infty}\frac{a_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n}\otimes_\alpha\sum_{n=0}^{\infty}\frac{b_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n}\\&=\sum_{n=0}^{\infty}\frac{1}{n!}\left(\sum_{m=0}^{n}\binom{n}{m}a_{n-m}b_m\right)\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n}.\end{aligned} \tag{7}$$

Definition 2.5 ([27])**:** If $0<\alpha\leq 1$, and x is a real number. The α-fractional exponential function is defined by

$$E_\alpha(x^\alpha)=\sum_{n=0}^{\infty}\frac{x^{n\alpha}}{\Gamma(n\alpha+1)}=\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_\alpha n}. \tag{8}$$

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On the other hand, the α-fractional cosine and sine function are defined as follows:

$$cos_\alpha(x^\alpha)=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n\alpha}}{\Gamma(2n\alpha+1)}=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha 2n}, \tag{9}$$

and

$$sin_\alpha(x^\alpha)=\sum_{n=0}^{\infty}\frac{(-1)^n x^{(2n+1)\alpha}}{\Gamma((2n+1)\alpha+1)}=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha (2n+1)}. \tag{10}$$

Definition 2.6 ([28])**:** If $0<\alpha\le 1$, and A is a matrix. The matrix α-fractional exponential function is defined by

$$E_\alpha(Ax^\alpha)=\sum_{n=0}^{\infty}A^n\frac{x^{n\alpha}}{\Gamma(n\alpha+1)}=\sum_{n=0}^{\infty}\frac{1}{n!}\left(A\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha n}. \tag{11}$$

And the matrix α-fractional cosine and sine function are defined as follows:

$$cos_\alpha(Ax^\alpha)=\sum_{n=0}^{\infty}A^n\frac{(-1)^n x^{2n\alpha}}{\Gamma(2n\alpha+1)}=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\left(A\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha 2n}, \tag{12}$$

and

$$sin_\alpha(Ax^\alpha)=\sum_{n=0}^{\infty}A^n\frac{(-1)^n x^{(2n+1)\alpha}}{\Gamma((2n+1)\alpha+1)}=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\left(A\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha (2n+1)}. \tag{13}$$

Theorem 2.7 (matrix fractional Euler's formula)**:** *If* $0<\alpha\le 1$, *and* A *is a real matrix, then*

$$E_\alpha(iAx^\alpha)=cos_\alpha(Ax^\alpha)+isin_\alpha(Ax^\alpha). \tag{14}$$

Theorem 2.8 (matrix fractional DeMoivre's formula)**:** *If* $0<\alpha\le 1$, p *is an integer, and* A *is a real matrix, then*

$$[cos_\alpha(Ax^\alpha)+isin_\alpha(Ax^\alpha)]^{\otimes_\alpha p}=cos_\alpha(pAx^\alpha)+isin_\alpha(pAx^\alpha). \tag{15}$$

Definition 2.9: The smallest positive real number T_α such that $E_\alpha(iT_\alpha)=1$, is called the period of $E_\alpha(ix^\alpha)$.

III. MAIN RESULTS

In this section, we obtain arbitrary order fractional derivative of two types of matrix fractional functions by using some techniques. At first, two lemmas are needed.

Lemma 3.1: *Suppose that* $0<\alpha\le 1$, r *is a real number, and* A *is a real matrix, then*

$$Ln_\alpha\big(1+rE_\alpha(iAx^\alpha)\big)=\frac{1}{2}Ln_\alpha(1+2rcos_\alpha(Ax^\alpha)+r^2)+i\cdot arctan_\alpha\left(rsin_\alpha(Ax^\alpha)\otimes_\alpha[1+rcos_\alpha(Ax^\alpha)]^{\otimes_\alpha(-1)}\right). \tag{16}$$

Proof $Ln_\alpha\big(1+rE_\alpha(iAx^\alpha)\big)$

$=Ln_\alpha\big(1+rcos_\alpha(Ax^\alpha)+irsin_\alpha(Ax^\alpha)\big)$ (by matrix fractional Euler's formula)

$$=Ln_\alpha\left(\begin{matrix}[1+2rcos_\alpha(Ax^\alpha)+r^2]^{\otimes_\alpha 1/2}\\ \otimes_\alpha\left[(1+rcos_\alpha(Ax^\alpha))\otimes_\alpha[1+2rcos_\alpha(Ax^\alpha)+r^2]^{\otimes_\alpha -1/2}+irsin_\alpha(Ax^\alpha)\otimes_\alpha[1+2rcos_\alpha(Ax^\alpha)+r^2]^{\otimes_\alpha -1/2}\right]\end{matrix}\right)$$

$$=Ln_\alpha\left([1+2rcos_\alpha(Ax^\alpha)+r^2]^{\otimes_\alpha 1/2}\otimes_\alpha E_\alpha\left(i\cdot arctan_\alpha\left(rsin_\alpha(Ax^\alpha)\otimes_\alpha[1+rcos_\alpha(Ax^\alpha)]^{\otimes_\alpha(-1)}\right)\right)\right)$$

$=\frac{1}{2}Ln_\alpha(1+2rcos_\alpha(Ax^\alpha)+r^2)+i\cdot arctan_\alpha\left(rsin_\alpha(Ax^\alpha)\otimes_\alpha[1+rcos_\alpha(Ax^\alpha)]^{\otimes_\alpha(-1)}\right)$. q.e.d.

Lemma 3.2: *If* $0<\alpha\le 1$, r *is a real number,* $|r|<1$, p *is a positive integer, and* A *is a real matrix, then*

$$\begin{aligned}&cos_\alpha(pAx^\alpha)\otimes_\alpha\frac{1}{2}Ln_\alpha(1+2rcos_\alpha(Ax^\alpha)+r^2)\\&-sin_\alpha(pAx^\alpha)\otimes_\alpha arctan_\alpha\left(rsin_\alpha(Ax^\alpha)\otimes_\alpha[1+rcos_\alpha(Ax^\alpha)]^{\otimes_\alpha(-1)}\right)\\&=\sum_{n=0}^{\infty}(-1)^n\frac{1}{n+1}r^{n+1}cos_\alpha((n+p+1)Ax^\alpha)\end{aligned} \tag{17}$$

And

$$cos_\alpha(pAx^\alpha)\otimes_\alpha arctan_\alpha\left(rsin_\alpha(Ax^\alpha)\otimes_\alpha[1+rcos_\alpha(Ax^\alpha)]^{\otimes_\alpha(-1)}\right)$$
$$+sin_\alpha(pAx^\alpha)\otimes_\alpha \tfrac{1}{2}Ln_\alpha(1+2rcos_\alpha(Ax^\alpha)+r^2)$$
$$=\sum_{n=0}^{\infty}(-1)^n\tfrac{1}{n+1}r^{n+1}sin_\alpha((n+p+1)Ax^\alpha) \quad (18)$$

Proof Since $|r|<1$, it follows that

$$[E_\alpha(iAx^\alpha)]^{\otimes_\alpha p}\otimes_\alpha Ln_\alpha(1+rE_\alpha(iAx^\alpha))$$
$$=E_\alpha(ipAx^\alpha)\otimes_\alpha \sum_{n=0}^{\infty}(-1)^n\tfrac{1}{n+1}[rE_\alpha(iAx^\alpha)]^{\otimes_\alpha(n+1)}$$
$$=E_\alpha(ipAx^\alpha)\otimes_\alpha \sum_{n=0}^{\infty}(-1)^n\tfrac{1}{n+1}r^{n+1}E_\alpha(i(n+1)Ax^\alpha) \quad \text{(by matrix fractional DeMoivre's formula)}$$
$$=\sum_{n=0}^{\infty}(-1)^n\tfrac{1}{n+1}r^{n+1}E_\alpha(i(n+p+1)Ax^\alpha)$$
$$=\sum_{n=0}^{\infty}(-1)^n\tfrac{1}{n+1}r^{n+1}cos_\alpha((n+p+1)Ax^\alpha)+i\cdot\sum_{n=0}^{\infty}(-1)^n\tfrac{1}{n+1}r^{n+1}sin_\alpha((n+p+1)Ax^\alpha)\,. \quad (19)$$

Therefore, by Lemma 3.1

$$cos_\alpha(pAx^\alpha)\otimes_\alpha \frac{1}{2}Ln_\alpha(1+2rcos_\alpha(Ax^\alpha)+r^2)$$
$$-sin_\alpha(pAx^\alpha)\otimes_\alpha arctan_\alpha\left(rsin_\alpha(Ax^\alpha)\otimes_\alpha[1+rcos_\alpha(Ax^\alpha)]^{\otimes_\alpha(-1)}\right)$$
$$=\sum_{n=0}^{\infty}(-1)^n\tfrac{1}{n+1}r^{n+1}cos_\alpha((n+p+1)Ax^\alpha),$$

and

$$arctan_\alpha\left(rsin_\alpha(Ax^\alpha)\otimes_\alpha[1+rcos_\alpha(Ax^\alpha)]^{\otimes_\alpha(-1)}\right)=\sum_{n=0}^{\infty}(-1)^n\tfrac{1}{n+1}r^{n+1}sin_\alpha((n+p+1)Ax^\alpha)\,.$$

q.e.d.

Theorem 3.3: *If* $0<\alpha\le 1$, r *is a real number,* $|r|<1, m, p$ *are positive integers, and* A *is a real matrix, then*

$$\left({}_0D_x^\alpha\right)^m\begin{bmatrix} cos_\alpha(pAx^\alpha)\otimes_\alpha \tfrac{1}{2}Ln_\alpha(1+2rcos_\alpha(Ax^\alpha)+r^2) \\ -sin_\alpha(pAx^\alpha)\otimes_\alpha arctan_\alpha\left(rsin_\alpha(Ax^\alpha)\otimes_\alpha[1+rcos_\alpha(Ax^\alpha)]^{\otimes_\alpha(-1)}\right)\end{bmatrix}$$
$$=\sum_{n=0}^{\infty}(-1)^n\tfrac{1}{n+1}r^{n+1}(n+p+1)^mA^m cos_\alpha\left((n+p+1)Ax^\alpha+m\cdot\tfrac{T_\alpha}{4}\right). \quad (20)$$

And

$$\left({}_0D_x^\alpha\right)^m\begin{bmatrix} cos_\alpha(pAx^\alpha)\otimes_\alpha arctan_\alpha\left(rsin_\alpha(Ax^\alpha)\otimes_\alpha[1+rcos_\alpha(Ax^\alpha)]^{\otimes_\alpha(-1)}\right) \\ +sin_\alpha(pAx^\alpha)\otimes_\alpha \tfrac{1}{2}Ln_\alpha(1+2rcos_\alpha(Ax^\alpha)+r^2)\end{bmatrix}$$
$$=\sum_{n=0}^{\infty}(-1)^n\tfrac{1}{n+1}r^{n+1}(n+p+1)^mA^m sin_\alpha\left((n+p+1)Ax^\alpha+m\cdot\tfrac{T_\alpha}{4}\right). \quad (21)$$

Proof By Lemma 3.2,

$$\left({}_0D_x^\alpha\right)^m\begin{bmatrix} cos_\alpha(pAx^\alpha)\otimes_\alpha \tfrac{1}{2}Ln_\alpha(1+2rcos_\alpha(Ax^\alpha)+r^2) \\ -sin_\alpha(pAx^\alpha)\otimes_\alpha arctan_\alpha\left(rsin_\alpha(Ax^\alpha)\otimes_\alpha[1+rcos_\alpha(Ax^\alpha)]^{\otimes_\alpha(-1)}\right)\end{bmatrix}$$

$$= \left({}_0D_x^\alpha\right)^m \left[\sum_{n=0}^{\infty}(-1)^n \frac{1}{n+1} r^{n+1} cos_\alpha((n+p+1)Ax^\alpha)\right]$$

$$= \sum_{n=0}^{\infty}(-1)^n \frac{1}{n+1} r^{n+1} \left({}_0D_x^\alpha\right)^m [cos_\alpha((n+p+1)Ax^\alpha)]$$

$$= \sum_{n=0}^{\infty}(-1)^n \frac{1}{n+1} r^{n+1} (n+p+1)^m A^m cos_\alpha\left((n+p+1)Ax^\alpha + m \cdot \frac{T_\alpha}{4}\right).$$

On the other hand,

$$\left({}_0D_x^\alpha\right)^m \left[\begin{array}{c} cos_\alpha(pAx^\alpha) \otimes_\alpha arctan_\alpha\left(rsin_\alpha(Ax^\alpha) \otimes_\alpha [1 + rcos_\alpha(Ax^\alpha)]^{\otimes_\alpha(-1)}\right) \\ +sin_\alpha(pAx^\alpha) \otimes_\alpha \frac{1}{2} Ln_\alpha(1 + 2rcos_\alpha(Ax^\alpha) + r^2) \end{array}\right]$$

$$= \left({}_0D_x^\alpha\right)^m \left[\sum_{n=0}^{\infty}(-1)^n \frac{1}{n+1} r^{n+1} sin_\alpha((n+p+1)Ax^\alpha)\right]$$

$$= \sum_{n=0}^{\infty}(-1)^n \frac{1}{n+1} r^{n+1} \left({}_0D_x^\alpha\right)^m [sin_\alpha((n+p+1)Ax^\alpha)]$$

$$= \sum_{n=0}^{\infty}(-1)^n \frac{1}{n+1} r^{n+1} (n+p+1)^m A^m sin_\alpha\left((n+p+1)Ax^\alpha + m \cdot \frac{T_\alpha}{4}\right).$$ q.e.d.

IV. CONCLUSION

In this paper, based on Jumarie's modified R-L fractional derivative and a new multiplication of fractional analytic functions, we can find arbitrary order fractional derivative of two types of matrix fractional functions. In addition, our results are generalizations of traditional calculus results. In the future, we will continue to use Jumarie type of R-L fractional calculus and the new multiplication of fractional analytic functions to solve problems in engineering mathematics and fractional differential equations.

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