ISSN 2350-1022

International Journal of Recent Research in Mathematics Computer Science and Information Technology
Vol. 11, Issue 2, pp: (19-22), Month: October 2024 – March 2025, Available at: www.paperpublications.org

Fractional Differential Problem of Some Type of Matrix Fractional Function

Chii-Huei Yu

School of Mathematics and Statistics, Zhaoqing University, Guangdong, China

DOI: https://doi.org/10.5281/zenodo.14169067

Published Date: 15-November-2024

Abstract: **In this paper, based on Jumarie type of Riemann-Liouville (R-L) fractional derivative and a new multiplication of fractional analytic functions, we can evaluate arbitrary order fractional derivative of some type of matrix fractional function. In fact, our result is a generalization of classical calculus result.**

Keywords: **Jumarie type of R-L fractional derivative, new multiplication, fractional analytic functions, matrix fractional function.**

I. INTRODUCTION

Fractional calculus is an extension of ordinary calculus, which has a history of more than 300 years. Fractional calculus with any real or complex derivative and integral originated from Euler's work, even earlier than Leibniz's work. In recent years, the application of fractional calculus in many different fields such as physics, mechanics, mathematical economics, viscoelasticity, biology, control theory, and electrical engineering [1-13].

However, the definition of fractional derivative is not unique. Commonly used definitions include Riemann-Liouville (R-L) fractional derivative, Caputo fractional derivative, Grunwald-Letnikov (G-L) fractional derivative, Jumarie's modified R-L fractional derivative [14-18]. Because Jumarie type of R-L fractional derivative helps to avoid non-zero fractional derivative of constant function, it is easier to use this definition to connect fractional calculus with classical calculus.

In this paper, based on Jumarie type of R-L fractional derivative and a new multiplication of fractional analytic functions, we study the fractional differential problem of the following type of matrix fractional function:

$$[E_\alpha(tAx^\alpha)]^{\otimes_\alpha p}\otimes_\alpha E_\alpha\left(E_\alpha(tAx^\alpha)\right),$$

where $0<\alpha\leq 1$, t is a real number, p is a positive integer, and A is a real matrix. Using some methods, we can evaluate arbitrary order fractional derivative of this type of matrix fractional function. In fact, our result is a generalization of ordinary calculus result.

II. PRELIMINARIES

At first, we introduce the fractional derivative used in this paper and its properties.

Definition 2.1 ([19])**:** Let $0<\alpha\leq 1$, and x_0 be a real number. The Jumarie type of Riemann-Liouville (R-L) α-fractional derivative is defined by

$$\left({}_{x_0}D_x^\alpha\right)[f(x)]=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{x_0}^{x}\frac{f(t)-f(x_0)}{(x-t)^\alpha}dt\ , \tag{1}$$

where $\Gamma(\ \ )$ is the gamma function. On the other hand, for any positive integer m, we define $\left({}_{x_0}D_x^\alpha\right)^m[f(x)]=\left({}_{x_0}D_x^\alpha\right)\left({}_{x_0}D_x^\alpha\right)\cdots\left({}_{x_0}D_x^\alpha\right)[f(x)]$, the m-th order α-fractional derivative of $f(x)$.

ISSN 2350-1022

International Journal of Recent Research in Mathematics Computer Science and Information Technology
Vol. 11, Issue 2, pp: (19-22), Month: October 2024 – March 2025, Available at: www.paperpublications.org

Proposition 2.2 ([20])**:** *If* α, β, x_0, C *are real numbers and* $\beta \geq \alpha > 0$, *then*

$$\left({}_{x_0}D_x^{\alpha}\right)\left[(x-x_0)^{\beta}\right] = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(x-x_0)^{\beta-\alpha}, \tag{2}$$

and

$$\left({}_{x_0}D_x^{\alpha}\right)[C] = 0. \tag{3}$$

Next, the definition of fractional analytic function is introduced.

Definition 2.3 ([21])**:** If x, x_0, and a_n are real numbers for all n, $x_0 \in (a,b)$, and $0 < \alpha \leq 1$. If the function $f_\alpha: [a,b] \to R$ can be expressed as an α-fractional power series, that is, $f_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}$ on some open interval containing x_0, then we say that $f_\alpha(x^\alpha)$ is α-fractional analytic at x_0. In addition, if $f_\alpha: [a,b] \to R$ is continuous on closed interval $[a,b]$ and it is α-fractional analytic at every point in open interval (a,b), then f_α is called an α-fractional analytic function on $[a,b]$.

In the following, we introduce a new multiplication of fractional analytic functions.

Definition 2.4 ([22])**:** If $0 < \alpha \leq 1$. Assume that $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are two α-fractional power series at $x = x_0$,

$$f_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}, \tag{4}$$

$$g_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{b_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}. \tag{5}$$

Then

$$\begin{aligned} & f_\alpha(x^\alpha) \otimes_\alpha g_\alpha(x^\alpha) \\ &= \sum_{n=0}^{\infty} \frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha} \otimes_\alpha \sum_{n=0}^{\infty} \frac{b_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha} \\ &= \sum_{n=0}^{\infty} \frac{1}{\Gamma(n\alpha+1)}\left(\sum_{m=0}^{n} \binom{n}{m} a_{n-m} b_m\right)(x-x_0)^{n\alpha}. \end{aligned} \tag{6}$$

Equivalently,

$$\begin{aligned} & f_\alpha(x^\alpha) \otimes_\alpha g_\alpha(x^\alpha) \\ &= \sum_{n=0}^{\infty} \frac{a_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n} \otimes_\alpha \sum_{n=0}^{\infty} \frac{b_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!}\left(\sum_{m=0}^{n} \binom{n}{m} a_{n-m} b_m\right)\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n}. \end{aligned} \tag{7}$$

Definition 2.5 ([23])**:** Let $0 < \alpha \leq 1$, and $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ be two α-fractional analytic functions defined on an interval containing x_0 ,

$$f_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha} = \sum_{n=0}^{\infty} \frac{a_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes n}, \tag{8}$$

$$g_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{b_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha} = \sum_{n=0}^{\infty} \frac{b_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes n}. \tag{9}$$

The compositions of $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are defined by

$$(f_\alpha \circ g_\alpha)(x^\alpha) = f_\alpha(g_\alpha(x^\alpha)) = \sum_{n=0}^{\infty} \frac{a_n}{n!}\left(g_\alpha(x^\alpha)\right)^{\otimes n}, \tag{10}$$

and

$$(g_\alpha \circ f_\alpha)(x^\alpha) = g_\alpha(f_\alpha(x^\alpha)) = \sum_{n=0}^{\infty} \frac{b_n}{n!}\left(f_\alpha(x^\alpha)\right)^{\otimes n}. \tag{11}$$

Definition 2.6 ([24])**:** If $0 < \alpha \leq 1$, and A is a matrix. The matrix α-fractional exponential function is defined by

$$E_\alpha(Ax^\alpha) = \sum_{n=0}^{\infty} A^n \frac{x^{n\alpha}}{\Gamma(n\alpha+1)} = \sum_{n=0}^{\infty} \frac{1}{n!}\left(A\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha n}. \quad (12)$$

III. MAIN RESULT

In this section, we evaluate arbitrary order fractional derivative of some type of matrix fractional function.

Theorem 3.1: *If* $0 < \alpha \leq 1$, t *is a real number,* m, p *are positive integers, and* A is a real matrix. *Then*

$$\left({}_0D_x^\alpha\right)^m \left[[E_\alpha(tAx^\alpha)]^{\otimes_\alpha p} \otimes_\alpha E_\alpha\left(E_\alpha(tAx^\alpha)\right)\right] = t^m A^m \sum_{n=0}^{\infty} \frac{1}{n!}(n+p)^m [E_\alpha((n+p)tAx^\alpha)]. \quad (13)$$

Proof $\left({}_0D_x^\alpha\right)^m \left[[E_\alpha(tAx^\alpha)]^{\otimes_\alpha p} \otimes_\alpha E_\alpha\left(E_\alpha(tAx^\alpha)\right)\right]$

$$= \left({}_0D_x^\alpha\right)^m \left[E_\alpha(ptAx^\alpha) \otimes_\alpha \sum_{n=0}^{\infty} \frac{1}{n!}\left(E_\alpha(tAx^\alpha)\right)^{\otimes_\alpha n}\right]$$

$$= \left({}_0D_x^\alpha\right)^m \left[E_\alpha(ptAx^\alpha) \otimes_\alpha \sum_{n=0}^{\infty} \frac{1}{n!} E_\alpha(ntAx^\alpha)\right]$$

$$= \left({}_0D_x^\alpha\right)^m \left[\sum_{n=0}^{\infty} \frac{1}{n!} E_\alpha((n+p)tAx^\alpha)\right]$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \left({}_0D_x^\alpha\right)^m [E_\alpha((n+p)tAx^\alpha)]$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} [(n+p)t]^m A^m E_\alpha((n+p)tAx^\alpha)$$

$$= t^m A^m \sum_{n=0}^{\infty} \frac{1}{n!}(n+p)^m [E_\alpha((n+p)tAx^\alpha)].$$

q.e.d.

IV. CONCLUSION

In this paper, based on Jumarie type of R-L fractional derivative and a new multiplication of fractional analytic functions, we can evaluate arbitrary order fractional derivative of some type of matrix fractional function by using some techniques. Moreover, our result is a generalization of classical calculus result. In the future, we will continue to use Jumarie's modified R-L fractional derivative and the new multiplication of fractional analytic functions to solve the problems in fractional differential equations and engineering mathematics.

REFERENCES

[1] V. E. Tarasov, Mathematical economics: application of fractional calculus, Mathematics, vol. 8, no. 5, 660, 2020.

[2] E. Soczkiewicz, Application of fractional calculus in the theory of viscoelasticity, Molecular and Quantum Acoustics vol.23, pp. 397-404, 2002.

[3] J. T. Machado, Fractional Calculus: Application in Modeling and Control, Springer New York, 2013.

[4] F. Mainardi, Fractional calculus: some basic problems in continuum and statistical mechanics, Fractals and Fractional Calculus in Continuum Mechanics, pp. 291-348, Springer, Wien, Germany, 1997.

[5] Mohd. Farman Ali, Manoj Sharma, Renu Jain, An application of fractional calculus in electrical engineering, Advanced Engineering Technology and Application, vol. 5, no. 2, pp, 41-45, 2016.

[6] R. Magin, Fractional calculus in bioengineering, part 1, Critical Reviews in Biomedical Engineering, vol. 32, no,1. pp.1-104, 2004.

[7] H. A. Fallahgoul, S. M. Focardi and F. J. Fabozzi, Fractional calculus and fractional processes with applications to financial economics, theory and application, Elsevier Science and Technology, 2016.

[8] M. F. Silva, J. A. T. Machado, and I. S. Jesus, Modelling and simulation of walking robots with 3 dof legs, in Proceedings of the 25th IASTED International Conference on Modelling, Identification and Control (MIC '06), pp. 271-276, Lanzarote, Spain, 2006.

[9] M. Teodor, Atanacković, Stevan Pilipović, Bogoljub Stanković, Dušan Zorica, Fractional Calculus with Applications in Mechanics: Vibrations and Diffusion Processes, John Wiley & Sons, Inc., 2014.

[10] C. -H. Yu, A study on fractional RLC circuit, International Research Journal of Engineering and Technology, vol. 7, no. 8, pp. 3422-3425, 2020.

[11] C. -H. Yu, A new insight into fractional logistic equation, International Journal of Engineering Research and Reviews, vol. 9, no. 2, pp.13-17, 2021.

[12] R. Hilfer (Ed.), Applications of Fractional Calculus in Physics, WSPC, Singapore, 2000.

[13] F. Duarte and J. A. T. Machado, Chaotic phenomena and fractional-order dynamics in the trajectory control of redundant manipulators, Nonlinear Dynamics, vol. 29, no. 1-4, pp. 315-342, 2002.

[14] I. Podlubny, Fractional Differential Equations, Academic Press, San Diego, Calif, USA, 1999.

[15] K. B. Oldham and J. Spanier, The Fractional Calculus, Academic Press, Inc., 1974.

[16] K. S. Miller and B. Ross, An introduction to the Fractional Calculus and Fractional Differential Equations, A Wiley-Interscience Publication, John Wiley & Sons, New York, USA, 1993.

[17] S. Das, Functional Fractional Calculus for System Identification and Control, 2nd ed., Springer-Verlag, Berlin, 2011.

[18] K. Diethelm, The Analysis of Fractional Differential Equations, Springer-Verlag, 2010.

[19] C. -H, Yu, Studying three types of matrix fractional integrals, International Journal of Interdisciplinary Research and Innovations, vol. 12, no. 4, pp. 35-39, 2024.

[20] C. -H, Yu, Evaluating fractional derivatives of two matrix fractional functions based on Jumarie type of Riemann-Liouville fractional derivative, International Journal of Engineering Research and Reviews, vol. 12, no. 4, pp. 39-43, 2024.

[21] C. -H, Yu, Study of fractional Fourier series expansions of two types of matrix fractional functions, International Journal of Mathematics and Physical Sciences Research, vol. 12, no. 2, pp. 13-17, 2024.

[22] C. -H, Yu, Fractional partial differential problem of some matrix two-variables fractional functions, International Journal of Mechanical and Industrial Technology, vol. 12, no. 2, pp. 6-13, 2024.

[23] C. -H, Yu, Study of two matrix fractional integrals by using differentiation under fractional integral sign, International Journal of Civil and Structural Engineering Research, vol. 12, no. 2, pp. 24-30, 2024.

[24] C. -H, Yu, Fractional differential problem of two matrix fractional hyperbolic functions, International Journal of Recent Research in Civil and Mechanical Engineering, vol. 11, no. 2, pp. 1-4, 2024.