Flexural Stability Analysis of Doubly Symmetric Single Cell Thin -Walled Box Column Based On Rayleigh- Ritz Method [RRM]

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Abstract: **This research work is aimed at determining the Flexural [F] critical buckling load, P^{Crit} for Doubly Symmetrical Single (DSS) cell Thin-Walled Columns [TWC] cross section at different boundary conditions using Rayleigh-Ritz Method (RRM) with Polynomial Shape Functions . It is the follow up of the works by Nwachukwu and others (2017) and Nwachukwu and others (2021a) where the governing equation for the Total Potential Energy Functional (TPEF) for a Thin- Walled Box Column (TWBC) applicable to RRM and peculiar TPEF for DSS cross – section were derived respectively. Using the derived DSS equations, series of stability matrices and polynomial equations were generated through the minimization of DSS TPEF for each of the Pinned-Pinned [S-S], Fixed-Pinned[C-S] , Fixed-Fixed [C-C] boundary conditions. Upon handy solution of the polynomials (stability matrices) with developed well-coordinated algorithm, flexural P^{Crit} values were obtained for the DSS cell cross- section and compared with works of Ezeh and Osadebe (2010) who used Vlasov method. The evaluated DSS –[F]-P^{Crit} values are found helpful in providing useful design data for selection of suitable TWC profile to any TWC designer with the assurance of meeting all engineering design safety criteria. Even though the RRM P^{Crit} values are smaller when compared with that of Vlasov's, it didn't fall short of TWC safety/stability criteria. Subsequently, it is recommended that structural designers should select design values that are a little bit greater than the DSS- RRM evaluated critical buckling values in order to ensure safety of the TWC structure.**

Keywords: **Stability / Buckling Analysis , DSS, Thin -Walled Box Column (TWBC) or Thin-Walled Column (TWC), Rayleigh- Ritz Method (RRM), Flexural Critical Buckling Load (P^{crit}).**

1. INTRODUCTION

A thin-walled structure (TWS), according to Murray (1984), is one which is made from thin plates joined along their edges. The plate thickness for the TWS however is small compared to other cross sectional dimensions which are in turn often small compared with the overall length of the member or structure. TWS have a high load-carrying capacity, despite their small thickness (Sudhir and others, 2014). Thin-walled columns (TWC) as well as other TWS are very light compared with alternative structures and therefore, they are used extensively in long-span bridges and other structures where weight and cost are prime considerations. They are widely used in the construction industry because of their light weight and economy, particularly for long span floors in industrial and public buildings and for storage structures for liquids and bulk materials, such as tanks, hoppers, silos, and coal houses. Thin- walled structures are especially suitable for use in constructing Aircrafts, exhibition pavilions, concert halls, and sports arenas because the variety of available shapes permits great architectural expression, the covering of wide areas, and flexibility in the choice of construction materials, including steel, aluminum, reinforced concrete, and laminated plastic. In a nutshell, according to Richard Lieu and others (1998), thin-plated structures are used extensively in building construction, automobile, aircraft, shipbuilding and other industries because of a number of favourable factors such as high strength-weight ratio, development of new materials and processes and the availability of efficient analytical methods.

Owing to the numerous applications of TWC or TWS in general and the resulted instability due their high carrying capacity, the study of stability becomes necessary. According to Srinath (2009), stability represents a fundamental problem in solid mechanics, which must be mastered to ensure the safety of structures against collapse. The theory of stability is of crucial importance for TWS applications to structural engineering, aerospace engineering, nuclear engineering, offshore, and ocean engineering. According to Bazant (2000), the theory of stability also plays an important role in certain problems of space structures, geotechnical structures, geophysics and material science. The continued importance and vitality of research on structural stability problem is due to technical and economic developments that demand the use of ever stronger and ever higher structures in an increasingly wider range of applications. According to Mohri and others (2008), such an expansion of use is made possible by developments in manufacturing, fabrication technology, computer- aided- design, economic competition and construction efficiency. These developments continually do not only change the way in which traditional structures are designed and built, but they also make possible the economic use of materials in other areas of application, such as offshore structures, transportation vehicles, and outer- space structures.

DSS are common examples of TWC cross –sections and according to Simao and Simoes da silva (2004), the use of very slender thin-walled cross-sections members have become increasingly in demand due to their high stiffness/weight ratio, in recent years. For about a century many branches of the industry have sought stronger and at the same time lighter structural solutions which optimize the effectiveness and the cost of the structures (Andreassen, 2012). Such industries cut across civil, offshore, mechanical, naval, and aerospace industries. Other cross- sections that the author has considered/ will consider include: Doubly Symmetric Single cell cross- section (DSS), Doubly Symmetric Multi- cell cross- section (DSM), Mono- Symmetric Single cell cross- section (MSS), Mono- Symmetric Multi- cell cross- section (MSM), Asymmetric Single cell cross- section (ASS) and Asymmetric Multi- cell cross- section (ASM).

This present study is an attempt to determine the flexural critical buckling load for a DSS TWC cross-section under different boundary conditions based on formulated RRM based TPEF. It is the follow up of the works by Nwachukwu and others (2017) and Nwachukwu and others (2021a) where the governing equations for the TPEF for a TWBC applicable to RRM and peculiar TPEF for DSS cross – section were derived respectively. Of recent, many researchers have carried out one form of analysis or the other on thin- walled box columns and related topics. For instant, Krolak and others (2009) presented a theoretical, numerical and experimental analysis of the stability and ultimate load of multi-cell thin-walled columns of rectangular and square cross-sections subjected to axial compression. Shanmugam and others (1989) presented a numerical method to investigate the ultimate strength behavior of thin-walled steel box columns subjected to axial loads and biaxial end moments. The work of Ezeh (2009) involved a theoretical formulation based on Vlasov's theory as modified by Varbanov, in analyzing flexural, flexural-torsional, and flexural-torsional-distortional buckling modes of thin-walled closed columns. Chidolue and Osadebe (2012), also used Vlasov's theory to carryout Torsional- Distortional analysis of thin-walled box girder bridges. Chidolue and Aginam (2012) investigated the effects of shape factor on the Flexural- Torsional-Distortional behavior of thin- walled box girder structures using Vlasov's Theory. Ezeh (2010) also investigated the buckling behavior of axially compressed multi- cell doubly symmetric thin- walled column using Vlasov's theory. The works of Osadebe and Chidolue (2012a), Osadebe and Chidolue (2012b), Osadebe and Ezeh (2009a), Osadebe and Ezeh (2009b) were also based on Vlasov's method. Again, Ezeh and Osadebe (2010) carried out a research work on the Comparative Study of Vlasov and Euler Instabilities Of Axially Compressed Thin-Walled Box Columns. Nwachukwu and others (2017) and Nwachukwu and others (2021a) derived the RRM based governing TPEF equation for the TWBC applicable to RRM and evaluate and formulate the peculiar TPEF for DSS cross – section respectively. Nwachukwu and others (2021b) evaluated and formulated the TPEF for DSM and MSM cross section. Again, Nwachukwu and others (2022a) have evaluated and formulated the peculiar TPEF for MSS TWC cross section. Finally, Nwachukwu and others (2022b) have also evaluated and formulated the peculiar TPEF for ASS TWC cross section. Thus in the area of stability analysis of thin-walled box (closed) columns, little or no effort has been done to use RRM with polynomial shape function to determine the flexural critical buckling load for a DSS cross-section. Henceforth, the need for this recent research work. The flexural DSS critical buckling values will form design data for TWC designers when selecting suitable profile for the TWC under stability design

2. THEORITICAL BACKGROUND ON DSS - RRM BASED TWBC FLEXURAL STABILTY ANALYSIS

2.1. THE FORMULATED GENERAL RRM - BASED TPEF FOR TWBC

The major objective of this work carried out by Nwachukwu and others (2017) was to formulate the general TPEF for the TWBC in line with RRM – based stability analysis. This general RRM based TPEF is shown in Eqn.(1).

$$\pi = k_1 \int_L v^2 x^2 (2L-x)^2 dx + k_2 \int_L (v^I)^2 dx + k_3 \int_L (v^{II})^2 dx - k_4 \int_L (v^I)^2 dx. \quad (1)$$

In Eqn.(1), $k_1 = \frac{Ap^2}{8EI^2}$; $k_2 = \frac{AG}{2}$; $k_3 = \frac{EI}{2}$; and $k_4 = \frac{P}{2}$ **2(a-d)**

Where P is critical buckling load, A is Cross sectional area, E is young modulus of elasticity, G is shear modulus, I is moment of inertia, and L is length of the thin- walled column.

Also from Eqn.(1), v = the displacement function, which is a function of polynomial shape function, ϕ

According to Rayleigh- Ritz Theory: $v = \sum_i^n c_i \phi_i = c_1\phi_1 + c_2\phi_2 + c_3\phi_3 + \ldots\ldots + c_n\phi_n$ **(3)** In Eqn.(3), c = undetermined coefficient / unknown constant and ϕ = Polynomial shape function. From the work of Nwachukwu and others (2021a). the Polynomial Shape Function, ϕ has been generated for the S-S, C-C and C-S boundary conditions.

2.2. THE FORMULATED PECULIAR RRM - BASED TPEF FOR DSS TWBC AT DIFFERENT BOUNDAY CONDITIONS

Again Nwachukwu and others (2021a) has used Eqn. (1) in combination with the generated polynomial shape function to formulate the peculiar/individual TPEF for Doubly Symmetric Single (DSS) cell TWBC for different boundary conditions as shown in Eqns. (5), (10) and (12).

(a). CASE 1: PINNED-PINNED(S-S)- DSS- TWBC.

$$\pi_{DSS}^{S-S} = k_1{}^{DSS} \varphi_1{}^{S-S} + k_2{}^{DSS} \varphi_2{}^{S-S} + k_3{}^{DSS} \varphi_3{}^{S-S} - k_4{}^{DSS} \varphi_4{}^{S-S} \quad (4)$$

$$= k_1{}^{DSS} \Big[24c_1{}^2 L^{10} - 60c_1{}^2 L^{11} + \frac{390c_1{}^2 L^{12}}{7} - 10c_1{}^2 L^{13} + \frac{10c_1{}^2 L^{14}}{3} - \frac{8c_1c_2\sqrt{6300}\,L^{10}}{5} + \frac{20c_1c_2\sqrt{6300}\,L^{11}}{3} - \frac{74c_1c_2\sqrt{6300}\,L^{12}}{7} + 8c_1c_2\sqrt{6300}\,L^{13} - \frac{26c_1c_2\sqrt{6300}\,L^{14}}{9} + \frac{2c_1c_2\sqrt{6300}\,L^{15}}{5} + 168c_2{}^2 L^{10} - 980c_2{}^2 L^{11} + 2310c_2{}^2 L^{12} - \frac{5565c_2{}^2 L^{13}}{2} + \frac{5390c_2{}^2 L^{14}}{3} - 588c_2{}^2 L^{15} + \frac{840c_2{}^2 L^{16}}{11} \Big] + k_2{}^{ASM} \Big[30c_1{}^2 - 60c_1{}^2 L + 40c_1{}^2 L^2 - 2c_1c_2\sqrt{6300} + 8c_1c_2\sqrt{6300}\,L - 12c_1c_2\sqrt{6300}\,L^2 + 6c_1c_2\sqrt{6300}\,L^3 + 210c_2{}^2 - 1260c_2{}^2 L + 3360c_2{}^2 L^2 - 3780c_2{}^2 L^3 + 1512c_2{}^2 L^4 \Big] + k_3{}^{DSS} \Big[\frac{120c_1{}^2}{L^2} - \frac{24c_1c_2\sqrt{6300}}{L^2} + \frac{24c_1c_2\sqrt{6300}}{L} + \frac{7560c_2{}^2}{L^2} - \frac{15120c_2{}^2}{L} + 10080c_2{}^2\Big] - k_4{}^{DSS} \Big[30c_1{}^2 - 60c_1{}^2 L + 40c_1{}^2 L^2 - 2c_1c_2\sqrt{6300} + 8c_1c_2\sqrt{6300}\,L - 12c_1c_2\sqrt{6300}\,L^2 + 6c_1c_2\sqrt{6300}\,L^3 + 210c_2{}^2 - 1260c_2{}^2 L + 3360c_2{}^2 L^2 - 3780c_2{}^2 L^3 + 1512c_2{}^2 L^4 \Big] \quad (5)$$

Where, $k_1{}^{DSS} = \frac{A^{DSS}p^2}{8EI^{2(DSS)}}$, $k_2{}^{DSS} = \frac{A^{DSS}G}{2}$, $k_3{}^{DSS} = \frac{EI^{DSS}}{2}$ & $k_4{}^{DSS} = \frac{P}{2}$ **6(a-d)**

From the works of Nwachukwu and others (2021a),

A^{DSS} = Cross- sectional Area in m^2 = 10at **(7)**

I^{DSS} = Moment of Inertia of the plate in m^4 = $0.83at^3 + 28a^3t$ **(8)**

t = Thickness of the column in metres

a = Column cross section dimension in metres

(b). CASE 2: FIXED-FIXED[C-C]- DSS- TWBC.

The peculiar TPEF for DSS-[C-C] TWBC has been obtained as follows:

$$\pi_{DSS}^{C-C} = k_1{}^{DSS}\,\varphi_1{}^{C-C} + k_2{}^{DSS}\,\varphi_2{}^{C-C} + k_3{}^{DSS}\varphi_3{}^{C-C} - k_4{}^{DSS}\varphi_4{}^{C-C} \qquad \textbf{(9)}$$

$$= k_1{}^{DSS}\,[360c_1{}^2\,L^{12} - 1575c_1{}^2L^{13} + 2870c_1{}^2L^{14} - 2772c_1{}^2L^{15} + \frac{16380c_1{}^2L^{16}}{11} - 420c_1{}^2L^{17} + \frac{630c_1{}^2L^{18}}{13} - \frac{72c_1c_2\sqrt{53900}\,L^{12}}{7} + 63c_1c_2\sqrt{53900}\,L^{13} - 162c_1c_2\sqrt{53900}\,L^{14} + \frac{2028c_1c_2\sqrt{53900}\,L^{15}}{10} - \frac{1836c_1c_2\sqrt{53900}\,L^{16}}{11} +$$
$$87c_1c_2\sqrt{53900}\,L^{17} - 24c_1c_2\sqrt{53900}\,L^{18} + \frac{36c_1c_2\sqrt{53900}\,L^{19}}{14} + 3960c_2{}^2\,L^{12} - 31185c_2{}^2L^{13} + 105490c_2{}^2\,L^{14} - 1995840c_2{}^2L^{15} + 230580c_2{}^2\,L^{16} - 166320c_2{}^2L^{17} + \frac{949410c_1{}^2L^{18}}{13} - 17820c_2{}^2L^{19} + 1848c_2{}^2L^{20}$$
$$+ k_2{}^{DSS}\,[840c_1{}^2L^2 - 3780c_1{}^2L^3 + 6552c_1{}^2L^4 - 5040c_1{}^2L^5 + 1440c_1{}^2L^6 - 24c_1c_2\sqrt{53900}L^2 + 171c_1c_2\sqrt{53900}L^3 - 432c_1c_2\sqrt{53900}L^4 + 564c_1c_2\sqrt{53900}L^5 - 360c_1c_2\sqrt{53900}L^6 + 90c_1c_2\sqrt{53900}L^7 + 9240c_2{}^2L^2 - 83160c_2{}^2L^3 + 310464c_2{}^2L^4 -$$
$$600600c_2{}^2L^5 + 633600c_2{}^2L^6 - 346500c_2{}^2L^7 + 77000c_2{}^2L^8] + k_3{}^{DSS}\,[\frac{2520c_1{}^2}{L^2} - \frac{8820c_1{}^2}{L} + 40320c_1{}^2 - 45360c_1{}^2L + 18144c_1{}^2L^2 - \frac{72c_1c_2\,\sqrt{53900}}{L^2} + \frac{648c_1c_2\,\sqrt{53900}}{L} - 2592c_1c_2\,\sqrt{53900} + 4896c_1c_2\,\sqrt{53900}\,L - 4320c_1c_2\,\sqrt{53900}L^2 +$$
$$1440c_1c_2\,\sqrt{53900}L^3 + \frac{27720c_2{}^2}{L^2} - \frac{332640c_2{}^2}{L} + 1884960c_2{}^2 - 5266800c_2{}^2L + 7650720c_2{}^2L^2 - 5544000c_2{}^2L^3 + 1584000c_2{}^2L^4\,] - k_4{}^{DSS}\,[840c_1{}^2L^2 - 3780c_1{}^2L^3 + 6552c_1{}^2L^4 - 5040c_1{}^2L^5 - 24c_1c_2\sqrt{53900}L^2 + 171c_1c_2\sqrt{53900}L^3 -$$
$$432c_1c_2\sqrt{53900}L^4 + 564c_1c_2\sqrt{53900}L^5 - 360c_1c_2\sqrt{53900}L^6 + 90c_1c_2\sqrt{53900}L^7 + 9240c_2{}^2L^2 - 83160c_2{}^2L^3 + 310464c_2{}^2L^4\; 600600c_2{}^2L^5 + 633600c_2{}^2L^6 - 346500c_2{}^2L^7 + 77000c_2{}^2L^8\;] \qquad \textbf{(10)}$$

Where $k_1{}^{DSS}$, $k_2{}^{DSS}$, $k_3{}^{DSS}$ and $k_4{}^{DSS}$ are defined in Eqns. 6(a-d) respectively.

(c). CASE 3: FIXED-PINNED[C-S]- DSS TWBC

The peculiar TPEF for DSS-[C-S] TWB has been obtained as follows:

$$\pi_{DSS}^{C-S} = k_1{}^{DSS}\,\varphi_1{}^{C-S} + k_2{}^{DSS}\,\varphi_2{}^{C-S} + k_3{}^{DSS}\varphi_3{}^{C-S} - k_4{}^{DSS}\varphi_4{}^{C-S} \qquad \textbf{(11)}$$

$$= k_1{}^{DSS}\,[\frac{22680c_1{}^2\,L^{12}}{133} - \frac{98280c_1{}^2\,L^{13}}{152} + \frac{174510c_1{}^2\,L^{14}}{171} - \frac{162540c_1{}^2\,L^{15}}{190} + \frac{83790c_1{}^2\,L^{16}}{209} - \frac{22680c_1{}^2\,L^{17}}{228} + \frac{2520c_1{}^2\,L^{18}}{247} -$$
$$\frac{216}{7}c_1c_2\frac{\sqrt{53900}}{\sqrt{4693}}L^{12} + \frac{10728}{8}c_1c_2\frac{\sqrt{53900}}{\sqrt{4693}}L^{13} - \frac{39078}{9}c_1c_2\frac{\sqrt{53900}}{\sqrt{4693}}L^{14} + \frac{58500}{10}c_1c_2\frac{\sqrt{53900}}{\sqrt{4693}}L^{15} - \frac{44640}{11}c_1c_2\frac{\sqrt{53900}}{\sqrt{4693}}L^{16} +$$
$$\frac{18000}{12}c_1c_2\frac{\sqrt{53900}}{\sqrt{4693}}L^{17} - \frac{3546}{13}c_1c_2\frac{\sqrt{53900}}{\sqrt{4693}}L^{18} + \frac{252}{14}c_1c_2\frac{\sqrt{53900}}{\sqrt{4693}}L^{19} + \frac{27720c_2{}^2\,L^{12}}{1729}$$
$$- \frac{2633400c_2{}^2\,L^{13}}{1976} + \frac{66784410c_2{}^2\,L^{14}}{2223} - \frac{203312340c_2{}^2\,L^{15}}{2470} + \frac{250637310c_2{}^2\,L^{16}}{2717} - \frac{151295760c_2{}^2\,L^{17}}{2964} + \frac{45952830c_2{}^2\,L^{18}}{3211} - \frac{6500340c_2{}^2\,L^{19}}{3458}$$
$$+ \frac{339570c_2{}^2\,L^{20}}{3705}\;] + k_2{}^{DSS}\,[\frac{22680c_1{}^2\,L^2}{57} - \frac{113400c_1{}^2\,L^3}{76)} + \frac{202230c_1{}^2\,L^4}{95} - \frac{151200c_1{}^2\,L^5}{114} + \frac{40320c_1{}^2\,L^6}{133} - \frac{216}{3}c_1c_2\frac{\sqrt{53900}}{\sqrt{4693}}L^2 +$$
$$\frac{15768}{4}c_1c_2\frac{\sqrt{53900}}{\sqrt{4693}}L^3 - \frac{61254}{5}c_1c_2\frac{\sqrt{53900}}{\sqrt{4693}}L^4 + \frac{81324}{6}c_1c_2\frac{\sqrt{53900}}{\sqrt{4693}}L^5 - \frac{39978}{7}c_1c_2\frac{\sqrt{53900}}{\sqrt{4693}}L^6 + \frac{5040}{8}c_1c_2\frac{\sqrt{53900}}{\sqrt{4693}}L^7 + \frac{27720c_2{}^2\,L^2}{741}$$
$$- \frac{3908520c_2{}^2\,L^3}{988} + \frac{143497970c_2{}^2\,L^4}{1235} - \frac{415273320c_2{}^2\,L^5}{1482} + \frac{379861020c_2{}^2\,L^6}{1729} - \frac{102841200c_2{}^2\,L^7}{1976} + \frac{8489250c_2{}^2\,L^8}{2223}] + k_3{}^{DSS}\,[\frac{22680c_1{}^2}{19L^2} -$$
$$\frac{226800c_1{}^2}{38L} + \frac{748440c_1{}^2}{57} - \frac{907200c_1{}^2\,L}{76} + \frac{362880c_1{}^2\,L^2}{95} - \frac{216}{L^2}c_1c_2\frac{\sqrt{53900}}{\sqrt{4693}} + \frac{31536}{2L}c_1c_2\frac{\sqrt{53900}}{\sqrt{4693}} - \frac{221832}{3}c_1c_2\frac{\sqrt{53900}}{\sqrt{4693}} +$$
$$\frac{480384}{4}c_1c_2\frac{\sqrt{53900}}{\sqrt{4693}}L - \frac{350352}{5}c_1c_2\frac{\sqrt{53900}}{\sqrt{4693}}L^2 + \frac{60480}{6}c_1c_2\frac{\sqrt{53900}}{\sqrt{4693}}L^3 + \frac{24640c_2{}^2}{247L^2} - \frac{7817040c_2{}^2}{494L} + \frac{568731240c_2{}^2}{741} - \frac{2489699520c_2{}^2\,L}{988}$$
$$+ \frac{3350350080c_2{}^2\,L^2}{1235} - \frac{123094400c_2{}^2\,L^3}{1482} + \frac{135828000c_2{}^2\,L^4}{1729}\,]$$

$$-k_4{}^{DSS}\,[\frac{22680c_1{}^2\,L^2}{57} - \frac{113400c_1{}^2\,L^3}{76)} + \frac{202230c_1{}^2\,L^4}{95} - \frac{151200c_1{}^2\,L^5}{114} + \frac{40320c_1{}^2\,L^6}{133} - \frac{216}{3}c_1c_2\frac{\sqrt{53900}}{\sqrt{4693}}L^2 + \frac{15768}{4}c_1c_2\frac{\sqrt{53900}}{\sqrt{4693}}L^3 -$$
$$\frac{61254}{5}c_1c_2\frac{\sqrt{53900}}{\sqrt{4693}}L^4 + \frac{81324}{6}c_1c_2\frac{\sqrt{53900}}{\sqrt{4693}}L^5 - \frac{39978}{7}c_1c_2\frac{\sqrt{53900}}{\sqrt{4693}}L^6 + \frac{5040}{8}c_1c_2\frac{\sqrt{53900}}{\sqrt{4693}}L^7$$
$$+ \frac{27720c_2{}^2\,L^2}{741} - \frac{3908520c_2{}^2\,L^3}{988} + \frac{143497970c_2{}^2\,L^4}{1235} - \frac{415273320c_2{}^2\,L^5}{1482} + \frac{379861020c_2{}^2\,L^6}{1729} - \frac{102841200c_2{}^2\,L^7}{1976} + \frac{8489250c_2{}^2\,L^8}{2223}\;] \qquad \textbf{(12)}$$

Where $k_1{}^{DSS}$, $k_2{}^{DSS}$, $k_3{}^{DSS}$ and $k_4{}^{DSS}$ are defined in Eqns.6 (a-d) respectively.

3. RRM FLEXURAL STABILITY ANALYSIS OF DOUBLY SYMMETRIC SINGLE CELL THIN-WALLED BOX COLUMN

The Total Potential Energy Functional for DSS cross section as derived by Nwachukwu and others (2021a) are stated in Eqns.(5), (10) and (12) respectively for Pinned- Pinned [S-S] , Fixed- Fixed [C-C] and Fixed- Pinned [C-S] Boundary conditions. For the flexural buckling analysis (that is for non- deformable DSS Thin- walled cross section), the K_2^{DSS} and K_3^{DSS} components disappear. For K_2^{DSS} is the torsional component of the buckling analysis, K_3^{DSS} is the distortional component (for deformed cross section) of the buckling analysis and K_1^{DSS} is the flexural component.

3.1. PINNED- PINNED [S-S]- DSS-TWC STABILITY ANALYSIS FOR FLEXURAL BUCKLING

Eqn. (5) is the TPEF for the Pinned- Pinned [S-S]- DSS- TWC .

Minimization of Eqn. (5) w r t C_1 and C_2 respectively gives Eqn.(13) in matrix form:

$$\begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \textbf{(13)}$$

Where,

$$\alpha_{11} = K_1^{DSS}\ L^{10}\ [48 - 120\,L + \tfrac{780}{7}L^2 - 20\,L^3 + \tfrac{20}{3}L^4\ ] + K_2^{DSS}\ [60 - 120L + 80L^2\ + K_3^{DSS}\ [240\ \tfrac{1}{L^2}]\ - K_4^{DSS}\ [60 - 120L + 80L^2\ ] \quad \textbf{(14)}$$

$$\alpha_{12} = K_1^{DSS}\ L^{10}\ [-\tfrac{8}{5}\sqrt{6300} + \tfrac{20}{3}\sqrt{6300}\,L - \tfrac{74}{7}\sqrt{6300}\,L^2 + 8\sqrt{6300}\,L^3 - \tfrac{26}{9}\sqrt{6300}\,L^4 + \tfrac{2}{5}\sqrt{6300}\,L^5\ ] + K_2^{DSS}\ [-2\sqrt{6300} + 8\sqrt{6300}\,L - 12\sqrt{6300}\,L^2 + 6\sqrt{6300}\,L^3\ ] + K_3^{DSS}\ [-24\tfrac{\sqrt{6300}}{L^2} + 24\tfrac{\sqrt{6300}}{L}\ ] - K_4^{DSS}\ [-2\sqrt{6300} + 8\sqrt{6300}\,L - 12\sqrt{6300}\,L^2 + 6\sqrt{6300}\,L^3\ ] \quad \textbf{(15)}$$

$$\alpha_{21} = K_1^{DSS}\ L^{10}\ [-\tfrac{8}{5}\sqrt{6300} + \tfrac{20}{3}\sqrt{6300}\,L - \tfrac{74}{7}\sqrt{6300}\,L^2 + 8\sqrt{6300}\,L^3 - \tfrac{26}{9}\sqrt{6300}\,L^4 + \tfrac{2}{5}\sqrt{6300}\,L^5\ ] + K_2^{DSS}\ [-2\sqrt{6300} + 8\sqrt{6300}\,L - 12\sqrt{6300}\,L^2 + 6\sqrt{6300}\,L^3\ ] + K_3^{DSS}\ [-24\tfrac{\sqrt{6300}}{L^2} + 24\tfrac{\sqrt{6300}}{L}\ ] - K_4^{DSS}\ [-2\sqrt{6300} + 8\sqrt{6300}\,L - 12\sqrt{6300}\,L^2 + 6\sqrt{6300}\,L^3\ ] \quad \textbf{(16)}$$

$$\alpha_{22} = K_1^{DSS}L^{10}\ [\ 336 - 1980\,L + 4620\,L^2 - 5565L^3 + \tfrac{10780}{3}L^4 - 1176L^5 + \tfrac{1680}{11}L^6\ ] + K_2^{DSS}\ [\ 420\ - 2520L + 6720\,L^2 - 7560\,L^3 + 3024\,L^4\ ] + K_3^{DSS}\ [\ 20160 + 15120\ \tfrac{1}{L^2} - 30240\tfrac{1}{L}\ ] - K_4^{DSS}\ [\ 420\ - 2520L + 6720\,L^2 - 7560\,L^3 + 3024\,L^4\ ] \quad \textbf{(17)}$$

For the non- trivial solution for C_1 *and* C_2 , the determinant of stability matrix in Eqn.(13) equals zero.

That is:
$$\begin{vmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{vmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (18)$$

Solving Eqn.(18) gives :

$$\alpha_{11}\ \alpha_{22} - \alpha_{12}\ \alpha_{21} = 0 \quad \textbf{(19)}$$

Then, for flexural buckling, $K_2^{DSS} = K_3^{DSS} = 0$ **(20)**

Thus, substituting Eqns. (14) - (17), noting Eqn.(20), we have;

$$\{\ K_1^{DSS}N_1^{SS} - K_4^{DSS}N_2^{SS}\}\ \{K_1^{DSS}N_3^{SS} - K_4^{DSS}N_4^{SS}\} - \{\ K_1^{DSS}N_5^{SS} - K_4^{DSS}N_6^{SS}\}\ \{K_1^{DSS}N_7^{SS} - K_4^{DSS}N_8^{SS}\} = 0 \quad \textbf{(21)}$$

$$\Rightarrow\ [K_1^{2(DSS)}N_1^{SS}N_3^{SS} - K_1^{DSS}K_4^{DSS}N_1^{SS}N_4^{SS} - K_1^{DSS}K_4^{DSS}N_2^{SS}N_3^{SS} + K_4^{2(DSS)}N_2^{SS}N_4^{SS}\ ] - [K_1^{2(DSS)}N_5^{SS}N_7^{SS} - K_1^{DSS}K_4^{DSS}N_5^{SS}N_8^{SS} - K_1^{DSS}K_4^{DSS}N_6^{SS}N_7^{SS} + K_4^{2(DSS)}N_6^{SS}N_8^{SS}] = 0 \quad \textbf{(22)}$$

$$\Rightarrow\ K_1^{2(DSS)}[N_1^{SS}N_3^{SS} - N_5^{SS}N_7^{SS}] - K_1^{DSS}K_4^{DSS}\ [N_1^{SS}N_4^{SS} + N_2^{SS}N_3^{SS} - N_5^{SS}N_8^{SS} - N_6^{SS}N_7^{SS}\ ] + K_4^{2(DSS)}\ [N_2^{SS}N_4^{SS} - N_6^{SS}N_8^{SS}] = 0 \quad \textbf{(23)}$$

Substituting the values of K_1^{DSS} and K_4^{DSS} in Eqns. 6(a- d) into Eqn.(23), we have:

$$\frac{P^4A^{2(DSS)}}{64E^2I^{4(DSS)}}\ [N_1^{SS}N_3^{SS} - N_5^{SS}N_7^{SS}] - \frac{P^3A^{(DSS)}}{16E\,I^{2(DSS)}}\ [N_1^{SS}N_4^{SS} + N_2^{SS}N_3^{SS} - N_5^{SS}N_8^{SS} - N_6^{SS}N_7^{SS}] + \frac{P^2}{4}\ [N_2^{SS}N_4^{SS} - N_6^{SS}N_8^{SS}] = 0 \quad \textbf{(24)}$$

Let $C^{K(DSS)} = \frac{A^{DSS}}{8E\, I^{2(DSS)}}$ = Kings Constant for DSS cross- section **(25)**

Substituting Eqn.(25) into Eqn.(24) yields :

$$P^4\theta_1^{DSS} - \frac{1}{2}P^3\theta_2^{DSS} + P^2\theta_3^{DSS} = 0 \qquad \textbf{(26)}$$

Where

$$\theta_1^{DSS} = C^{K^{2(DSS)}}[N_1^{SS}N_3^{SS} - N_5^{SS}N_7^{SS}] \qquad \textbf{(27)}$$

$$\theta_2^{DSS} = C^{K(DSS)}[N_1^{SS}N_4^{SS} + N_2^{SS}N_3^{SS} - N_5^{SS}N_8^{SS} - N_6^{SS}N_7^{SS}] \qquad \textbf{(28)}$$

$$\theta_3^{DSS} = \frac{1}{4}[N_2^{SS}N_4^{SS} - N_6^{SS}N_8^{SS}] \qquad \textbf{(29)}$$

$$N_1^{SS} = L^{10}\,[48 - 120\,L + \frac{780}{7}L^2 - 20\,L^3 + \frac{20}{3}L^4] \qquad \textbf{(30)}$$

$$N_2^{SS} = [60 - 120L + 80L^2] \qquad \textbf{(31)}$$

$$N_3^{SS} = L^{10}\,[\,336 - 1980\,L + 4620\,L^2 - 5565L^3 + \frac{10780}{3}L^4 - 1176L^5 + \frac{1680}{11}L^6\,] \qquad \textbf{(32)}$$

$$N_4^{SS} = 420 - 2520L + 6720\,L^2 - 7560\,L^3 + 3024\,L^4 \qquad \textbf{(33)}$$

$$N_5^{SS} = L^{10}\,[-\frac{8}{5}\sqrt{6300} + \frac{20}{3}\sqrt{6300}\,L - \frac{74}{7}\sqrt{6300}\,L^2 + 8\sqrt{6300}\,L^3 - \frac{26}{9}\sqrt{6300}\,L^4 + \frac{2}{5}\sqrt{6300}\,L^5] \qquad \textbf{(34)}$$

$$N_6^{SS} = -2\sqrt{6300} + 8\sqrt{6300}\,L - 12\sqrt{6300}\,L^2 + 6\sqrt{6300}\,L^3 \qquad \textbf{(35)}$$

$$N_7^{SS} = L^{10}\,[-\frac{8}{5}\sqrt{6300} + \frac{20}{3}\sqrt{6300}\,L - \frac{74}{7}\sqrt{6300}\,L^2 + 8\sqrt{6300}\,L^3 - \frac{26}{9}\sqrt{6300}\,L^4 + \frac{2}{5}\sqrt{6300}\,L^5] \qquad \textbf{(36)}$$

$$N_8^{SS} = -2\sqrt{6300} + 8\sqrt{6300}\,L - 12\sqrt{6300}\,L^2 + 6\sqrt{6300}\,L^3 \qquad \textbf{(37)}$$

From Eqn.(26), we have: $P^2\,(P^2\theta_1^{DSS} - \frac{1}{2}P\theta_2^{DSS} + \theta_3^{DSS}) = 0$ **(38)**

Since $P^2 \neq 0$, $\Longrightarrow P^2\theta_1^{DSS} - \frac{1}{2}P\theta_2^{DSS} + \theta_3^{DSS} = 0$ or $2P^2\theta_1^{DSS} - P\theta_2^{DSS} + 2\,\theta_3^{DSS} = 0$ **39(a- c)**

Where

P or P_{DSS}^{F-SS} or P^{crit} is the critical buckling load for Flexural –DSS-SS-TWC stability analysis

3.2. FIXED- FIXED [C-C]- DSS-TWC STABILITY ANALYSIS FOR FLEXURAL BUCKLING

Eqn. (10) is the TPEF for the Fixed- Fixed [C-C]- DSS- TWC .

Minimization of Eqn. (10) w r t C_1 and C_2 respectively gives Eqn.(40) in matrix form:

$$\begin{bmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad \textbf{(40)}$$

Where

$\beta_{11} = K_1^{DSS}\,L^{12}[720 - 3150L + 5740\,L^2 - 5544\,L^3 + \frac{32760}{11}L^4 - 840\,L^5 + \frac{1260}{13}L^6] + K_2^{DSS}L^2\,[1680 - 7560L + 13104\,L^2 - 10080\,L^3 + 2880L^4\,] + K_3^{DSS}\,[5040\,\frac{1}{L^2} - 17640\,\frac{1}{L} + 80640 - 90720L + 36288L^2] - K_4^{DSS}L^2\,[1680 - 7560L + 13104\,L^2 - 10080\,L^3 + 2880L^4\,]$ **(41)**

$\beta_{12} = K_1^{DSS}\,L^{12}\,[-\frac{72}{7}\sqrt{53900} + 63\sqrt{53900}\,L - 162\sqrt{53900}\,L^2 + \frac{2028}{10}\sqrt{53900}\,L^3 - \frac{1836}{11}\sqrt{53900}\,L^4 + 87\sqrt{53900}\,L^5 - 24\sqrt{53900}\,L^6 + \frac{36}{14}\sqrt{53900}\,L^7] + K_2^{DSS}L^2\,[-24\sqrt{53900} + 171\sqrt{53900}\,L - 432\sqrt{53900}\,L^2 + 564\sqrt{53900}\,L^3 - 360\sqrt{53900}\,L^4 + 90\sqrt{53900}\,L^5\,] + K_3^{DSS}\,[-\frac{72}{L^2}\sqrt{53900} + \frac{648}{L}\sqrt{53900} - 2592\sqrt{53900} + 4896\sqrt{53900} - 4320\sqrt{6300}\,L^2 + 1440\sqrt{53900}\,L^3] - K_4^{DSS}L^2\,[-24\sqrt{53900} + 171\sqrt{53900}\,L - 432\sqrt{53900}\,L^2 + 564\sqrt{53900}\,L^3 - 360\sqrt{53900}\,L^4 + 90\sqrt{53900}\,L^5\,]$ **(42)**

$\beta_{21} = K_1^{DSS} L^{12}$ [- $\frac{72}{7}\sqrt{53900}$ + 63 $\sqrt{53900}\,L$ – 162 $\sqrt{53900}\,L^2$ + $\frac{2028}{10}\sqrt{53900}\,L^3$ - $\frac{1836}{11}\sqrt{53900}\,L^4$ + 87 $\sqrt{53900}\,L^5$ - 24 $\sqrt{53900}\,L^6$ + $\frac{36}{14}\sqrt{53900}\,L^7$] + $K_2^{DSS} L^2$ [- 24 $\sqrt{53900}$ + 171 $\sqrt{53900}\,L$ - 432 $\sqrt{53900}\,L^2$ + 564 $\sqrt{53900}\,L^3$ - 360 $\sqrt{53900}\,L^4$ + 90 $\sqrt{53900}\,L^5$] + K_3^{DSS} [- $\frac{72}{L^2}\sqrt{53900}$ + $\frac{648}{L}\sqrt{53900}$ – 2592 $\sqrt{53900}$ + 4896 $\sqrt{53900}\,L$ –4320 $\sqrt{53900}\,L^2$ + 1440 $\sqrt{53900}\,L^3$] - $K_4^{DSS} L^2$[- 24$\sqrt{53900}$ + 171 $\sqrt{53900}\,L$ - 432 $\sqrt{53900}\,L^2$ + 564 $\sqrt{53900}\,L^3$ - 360 $\sqrt{53900}\,L^4$ + 90 $\sqrt{53900}\,L^5$] **(43)**

And

$\beta_{22} = K_1^{DSS} L^{12}$ [7920 – 62370 L + 210980 L^2 -3991680 L^3+ 461160 L^4 - 332640 L^5 + $\frac{1898820}{13} L^6$ - 35640 L^7 + 3696 L^8] + $K_2^{DSS} L^2$ [18480 - 166320 L + 620928 L^2 - 1201200 L^3 + 1267200 L^4 - 693000 L^5 +154000 L^6] + K_3^{DSS} [55440 $\frac{1}{L^2}$ - 665280 $\frac{1}{L}$ + 3769920 - 10533600L + 15301440 L^2 - 11088000L^3 + 3168000L^4] - $K_4^{DSS} L^2$ [18480 - 166320 L + 620928 L^2 - 1201200 L^3 + 1267200 L^4 - 693000 L^5 +154000 L^6] **(44)**

For the non- trivial solution for C_1 *and* C_2 , the determinant of stability matrix of Eqn.(40) equals zero.

$$\begin{vmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{vmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad \textbf{(45)}$$

Solving Eqn.(45) yields: $\beta_{11}\ \beta_{22}$ - $\beta_{12}\,\beta_{21}$ = 0 **(46)**

Since we are considering only flexural buckling $\Rightarrow$ K_2^{DSS} = K_3^{DSS} = 0 **(47)**

Now, substituting Eqns. (41) - (44), noting Eqn.(47), we have;

$\{ K_1^{DSS} N_1^{CC} - K_4^{DSS} N_2^{CC}\} \{K_1^{DSS} N_3^{CC} - K_4^{DSS} N_4^{CC}\}$ - $\{ K_1^{DSS} N_5^{CC} - K_4^{DSS} N_6^{CC}\} \{K_1^{DSS} N_7^{CC} - K_4^{DSS} N_8^{CC}\}$ = 0 **(48)**

$\Rightarrow$ $[K_1^{2(DSS)} N_1^{CC} N_3^{CC} - K_1^{DSS} K_4^{DSS} N_1^{CC} N_4^{CC} - K_1^{DSS} K_4^{DSS} N_2^{CC} N_3^{CC} + K_4^{2(DSS)} N_2^{CC} N_4^{CC}]$–$[K_1^{2(DSS)} N_5^{CC} N_7^{CC} - K_1^{DSS} K_4^{DSS} N_5^{CC} N_8^{CC} - K_1^{DSS} K_4^{DSS} N_6^{CC} N_7^{CC} + K_4^{2(DSS)} N_6^{CC} N_8^{CC}]$ = 0 **(49)**

$\Rightarrow$ $K_1^{2(DSS)}[N_1^{CC} N_3^{CC} - N_5^{CC} N_7^{CC}]$ - $K_1^{DSS} K_4^{DSS}$ $[N_1^{CC} N_4^{CC} + N_2^{CC} N_3^{CC} - N_5^{CC} N_8^{CC} - N_6^{CC} N_7^{CC}]$ + $K_4^{2(DSS)}[N_2^{CC} N_4^{CC} - N_6^{CC} N_8^{CC}]$ = 0 **(50)**

Substituting the values of K_1^{DSS} and K_4^{DSS}in Eqns. 6(a- d) into Eqn.(23), we have::

$\frac{P^4 A^{2(DSS)}}{64E^2 I^{4(DSS)}} [N_1^{CC} N_3^{CC} - N_5^{CC} N_7^{CC}]$ - $\frac{P^3 A^{(DSS)}}{16E\, I^{2(DSS)}} [N_1^{CC} N_4^{CC} + N_2^{CC} N_3^{CC} - N_5^{CC} N_8^{CC} - N_6^{CC} N_7^{CC}]$ + $\frac{P^2}{4} [N_2^{CC} N_4^{CC} - N_6^{CC} N_8^{CC}]$ **(51)**

Eqn.(51) can be further simplified to:

$$P^4\theta_1^{DSS-CC} - \tfrac{1}{2}P^3\theta_2^{DSS-CC} + P^2\theta_3^{DSS-CC} = 0 \qquad \textbf{(52)}$$

$$\Rightarrow P^2\,(P^2\theta_1^{DSS-CC} - \tfrac{1}{2}P\theta_2^{DSS-CC} + \theta_3^{DSS-CC}) = 0 \qquad \textbf{(53)}$$

Since $P^2 \neq 0$, then, $P^2\theta_1^{DSS-CC} - \frac{1}{2}P\theta_2^{DSS-CC} + \theta_3^{DSS-CC} = 0$ or $2P^2\theta_1^{DSS-CC} - P\theta_2^{DSS-CC} + 2\,\theta_3^{DSS-CC}$ **54 (a-b)**

Where

$$\theta_1^{DSS-CC} = C^{K^{2(DSS)}}[N_1^{CC} N_3^{CC} - N_5^{CC} N_7^{CC}] \qquad \textbf{(55)}$$

$$\theta_2^{DSS-CC} = C^{K^{(DSS)}}[N_1^{CC} N_4^{CC} + N_2^{CC} N_3^{CC} - N_5^{CC} N_8^{CC} - N_6^{CC} N_7^{CC}] \qquad \textbf{(56)}$$

$$\theta_3^{DSS-CC} = \tfrac{1}{4}[N_2^{CC} N_4^{CC} - N_6^{CC} N_8^{CC}] \qquad \textbf{(57)}$$

$$C^{K^{(DSS)}} = \frac{A^{DSS}}{8E\, I^{2(DSS)}} \qquad \textbf{(58)}$$

$$N_1^{CC} = L^{12}[720 - 3150L + 5740\,L^2 - 5544\,L^3 + \tfrac{32760}{11}L^4 - 840\,L^5 + \tfrac{1260}{13}L^6] \qquad \textbf{(59)}$$

$$N_2^{CC} = L^2\,[1680 - 7560L + 13104\,L^2 - 10080\,L^3 + 2880L^4\,] \qquad \textbf{(60)}$$

$N_3^{CC} = L^{12}$ [7920 – 62370 L + 210980 L^2 -3991680 L^3 + 461160 L^4 - 332640 L^5 + $\frac{1898820}{13}L^6$ - 35640 L^7 + 3696 L^8] **(61)**

$N_4^{CC} = L^2$ [18480 - 166320 L + 620928 L^2 - 1201200 L^3 + 1267200 L^4 - 693000 L^5 +154000 L^6] **(62)**

$N_5^{CC} = L^{12}$ [- $\frac{72}{7}\sqrt{53900}$ + 63 $\sqrt{53900}\,L$ – 162 $\sqrt{53900}\,L^2$ + $\frac{2028}{10}\sqrt{53900}\,L^3$ - $\frac{1836}{11}\sqrt{53900}\,L^4$ + 87 $\sqrt{53900}\,L^5$ - 24 $\sqrt{53900}\,L^6$ + $\frac{36}{14}\sqrt{53900}\,L^7$] **(63)**

$N_6^{CC} = L^2$ [- 24 $\sqrt{53900}$ + 171 $\sqrt{53900}\,L$ - 432 $\sqrt{53900}\,L^2$ + 564 $\sqrt{53900}\,L^3$ - 360 $\sqrt{53900}\,L^4$ + 90 $\sqrt{53900}\,L^5$] **(64)**

$N_7^{CC} = L^{12}$ [- $\frac{72}{7}\sqrt{53900}$ + 63 $\sqrt{53900}\,L$ – 162 $\sqrt{53900}\,L^2$ + $\frac{2028}{10}\sqrt{53900}\,L^3$ - $\frac{1836}{11}\sqrt{53900}\,L^4$ + 87 $\sqrt{53900}\,L^5$ - 24 $\sqrt{53900}\,L^6$ + $\frac{36}{14}\sqrt{53900}\,L^7$] **(65)**

$N_8^{CC} = L^2$ [- 24 $\sqrt{53900}$ + 171 $\sqrt{53900}\,L$ - 432 $\sqrt{53900}\,L^2$+ 564 $\sqrt{53900}\,L^3$ - 360 $\sqrt{53900}\,L^4$+ 90 $\sqrt{53900}\,L^5$] **(66)**

Where

P or P_{DSS}^{F-CC} is the critical buckling load for Flexural –DSS-CC-TWC stability analysis

3.3. FIXED- PINNED [C-S]- DSS-TWC STABILITY ANALYSIS FOR FLEXURAL BUCKLING

Eqn. (12) is the TPEF for the Fixed-Pinned [C-S]- DSS- TWC .

Minimization of Eqn. (12) w r t C_1 and C_2 respectively gives Eqn.(67) in matrix form:

$$\begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad \textbf{(67)}$$

Where

$y_{11} = K_1^{DSS}\, L^{12}\,[\frac{45360}{133} - \frac{196560}{152}L + \frac{349020}{171}L^2 - \frac{325080}{190}L^3 + \frac{167580}{209}L^4 - \frac{45360}{228}L^5 + \frac{5040}{247}L^6] + K_2^{DSS}\, L^2[\frac{45260}{57} - \frac{226800}{76}L + \frac{404460}{95}L^2 - \frac{302400}{114}L^3 + \frac{80640}{133}L^4] + K_3^{DSS}\,[\frac{45360}{19}\frac{1}{L^2} - \frac{453600}{38}\frac{1}{L} + \frac{1496880}{57} - \frac{1814400}{76}L + \frac{725760}{95}L^2] - K_4^{DSS}\, L^2[\frac{45360}{57} - \frac{226800}{76}L + \frac{404460}{95}L^2 - \frac{302400}{114}L^3 + \frac{80640}{133}L^4]$ **(68)**

$y_{12} = K_1^{DSS}\, L^{12}\,[-\frac{216}{7}M_3 + \frac{10728}{8}M_3L - \frac{39078}{9}M_3L^2 + \frac{58500}{10}M_3L^3 - \frac{44640}{11}M_3L^4 + \frac{18000}{12}M_3L^4 - \frac{3546}{13}M_3L^6 + \frac{252}{14}M_3L^7 + K_2^{DSS}\, L^2[-\frac{216}{3}M_3 + \frac{15768}{4}M_3L - \frac{61254}{5}M_3L^2 + \frac{81324}{6} - \frac{39978}{7}M_3L^4 + \frac{5040}{8}M_3L^6] + K_3^{DSS}\,[-\frac{216}{L^2} + \frac{31536}{2L}M_3 - \frac{221832}{3}M_3 + \frac{480384}{4}M_3L - \frac{350352}{5}M_3L^2 + \frac{60480}{6}M_3L^3] - K_4^{DSS}\, L^2[-\frac{216}{3}M_3 + \frac{15768}{4}M_3L - \frac{61254}{5}M_3L^2 + \frac{81324}{6}M_3L^3 - \frac{39978}{7}M_3L^4 + \frac{5040}{8}M_3L^5]$ **(69)**

$y_{21} = K_1^{DSS}\, L^{12}[-\frac{216}{7}M_3 + \frac{10728}{8}M_3L - \frac{39078}{9}M_3L^2 + \frac{58500}{10}M_3L^3 - \frac{44640}{11}M_3L^4 + \frac{18000}{12}M_3L^5 - \frac{3546}{13}M_3L^6 + \frac{252}{14}M_3L^7] + K_2^{DSS}L^2\,[-\frac{216}{3}M_3 + \frac{15768}{4}M_3L - \frac{61254}{5}M_3L^2 + \frac{81324}{6}M_3L^3 - \frac{39978}{7}M_3L^4 + \frac{5040}{8}M_3L^5] + K_3^{DSS}\,[-\frac{216}{L^2}M_3 + \frac{31536}{2L}M_3 - \frac{221832}{3}M_3 + \frac{480384}{4}M_3L - \frac{350352}{5}M_3L^2 + \frac{60480}{6}M_3L^3] - K_4^{DSS}L^2\,[-\frac{216}{3}M_3 + \frac{15768}{4}M_3L - \frac{61254}{5}M_3L^2 + \frac{81324}{6}M_3L^3 - \frac{39978}{7}M_3L^4 + \frac{5040}{8}M_3L^5]$ **(70)**

And

$y_{22} = K_1^{DSS}\, L^{12}[\frac{55440}{1729} - \frac{5266800}{1976}L + \frac{133568820}{2223}L^2 - \frac{406624680}{2470}L^3 + \frac{501274620}{2717}L^4 - \frac{3025915200}{2964}L^5 + \frac{91905660}{3211}L^6 - \frac{13000680}{3458}L^7 + \frac{679140}{3705}L^8 + K_2^{DSS}L^2[\frac{55440}{741} - \frac{7817040}{988}L + \frac{2869959400}{1235}L^2 - \frac{830546640}{1482}L^3 + \frac{759722040}{1729}L^4 - \frac{205682400}{1976}L^5 + \frac{16978500}{2223}L^6] + K_3^{DSS}\,[\frac{49280}{247}\frac{1}{L^2} - \frac{15634080}{494L}\frac{1}{L} + \frac{1137462480}{741} - \frac{4979399040}{988}L + \frac{6700700160}{1235}L^2 - \frac{246188800}{1482}L^3 + \frac{27165600}{1729}L^4] - K_4^{DSS}L^2\,[-\frac{55440}{741} - \frac{7817040}{988}L + \frac{2869959400}{1235}L^2 - \frac{830546640}{1482}L^3 + \frac{759722040}{1729}L^4 - \frac{205682400}{1976}L^5 + \frac{16978500}{2223}L^6]$ **(71)**

Where $M_3 = \sqrt{\frac{53900}{4693}}$ **(72)**

For the non- trivial solution for C_1 *and* C_2 , the determinant of stability matrix of Eqn.(67) equals zero.

$$\begin{vmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{vmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \textbf{(73)}$$

Solving Eqn.(73) yields: $y_{11}\, y_{22} - y_{12}\, y_{21} = 0$ **(74)**

Since we are considering only flexural buckling, $\Longrightarrow K_2^{DSS} = K_3^{DSS} = 0$ **(75)**

Now, substituting Eqns. (68) - (71), noting Eqn.(75), we have;

$$\{ K_1^{DSS}N_1^{CS} - K_4^{DSS}N_2^{CS}\} \{K_1^{DSS}N_3^{CS} - K_4^{DSS}N_4^{CS}\} - \{ K_1^{DSS}N_5^{CS} - K_4^{DSS}N_6^{CS}\} \{K_1^{DSS}N_7^{CS} - K_4^{DSS}N_8^{CS}\} = 0 \quad \textbf{(76)}$$

Where

$$N_1^{CS} = L^{12} [\tfrac{45360}{133} - \tfrac{196560}{152}L + \tfrac{349020}{171}L^2 - \tfrac{325080}{190}L^3 + \tfrac{167580}{209}L^4 - \tfrac{45360}{228}L^5 + \tfrac{5040}{247}L^6] \quad \textbf{(77)}$$

$$N_2^{CS} = L^2[\tfrac{45260}{57} - \tfrac{226800}{76}L + \tfrac{404460}{95}L^2 - \tfrac{302400}{114}L^3 + \tfrac{80640}{133}L^4] \quad \textbf{(78)}$$

$$N_3^{CS} = L^{12}[\tfrac{55440}{1729} - \tfrac{5266800}{1976}L + \tfrac{133568820}{2223}L^2 - \tfrac{406624680}{2470}L^3 + \tfrac{501274620}{2717}L^4 - \tfrac{3025915200}{2964}L^5 + \tfrac{91905660}{3211}L^6 - \tfrac{13000680}{3458}L^7 + \tfrac{679140}{3705}L^8] \quad \textbf{(79)}$$

$$N_4^{CS} = L^2[\tfrac{55440}{741} - \tfrac{7817040}{988}L + \tfrac{2869959400}{1235}L^2 - \tfrac{830546640}{1482}L^3 + \tfrac{759722040}{1729}L^4 - \tfrac{205682400}{1976}L^5 + \tfrac{16978500}{2223}L^6] \quad \textbf{(80)}$$

$$N_5^{CS} = L^{12} [- \tfrac{216}{7}M_3 + \tfrac{10728}{8}M_3L - \tfrac{39078}{9}M_3L^2 + \tfrac{58500}{10}M_3L^3 - \tfrac{44640}{11}M_3L^4 + \tfrac{18000}{12}M_3L^4 - \tfrac{3546}{13}M_3L^6 + \tfrac{252}{14}M_3L^7] \quad \textbf{(81)}$$

$$N_6^{CS} = L^2[- \tfrac{216}{3}M_3 + \tfrac{15768}{4}M_3L - \tfrac{61254}{5}M_3L^2 + \tfrac{81324}{6}M_3L^3 - \tfrac{39978}{7}M_3L^4 + \tfrac{5040}{8}M_3L^6] \quad \textbf{(82)}$$

$$N_7^{CS} = L^{12} [- \tfrac{216}{7}M_3 + \tfrac{10728}{8}M_3L - \tfrac{39078}{9}M_3L^2 + \tfrac{58500}{10}M_3L^3 - \tfrac{44640}{11}M_3L^4 + \tfrac{18000}{12}M_3L^4 - \tfrac{3546}{13}M_3L^6 + \tfrac{252}{14}M_3L^7] \quad \textbf{(83)}$$

$$N_8^{CS} = L^2[- \tfrac{216}{3}M_3 + \tfrac{15768}{4}M_3L - \tfrac{61254}{5}M_3L^2 + \tfrac{81324}{6}M_3L^3 - \tfrac{39978}{7}M_3L^4 + \tfrac{5040}{8}M_3L^6] \quad \textbf{(84)}$$

Simplifying Eqn.(76) further, we have :

$$K_1^{2(DSS)}[N_1^{CS}N_3^{CS} - N_5^{CS}N_7^{CS}] - K_1^{DSS}K_4^{DSS} [N_1^{CS}N_4^{CS} + N_2^{CS}N_3^{CS} - N_5^{CS}N_8^{CS} - N_6^{CS}N_7^{CS}] + K_4^{2(DSS)}[N_2^{CS}N_4^{CS} - N_6^{CS}N_8^{CS}] = 0 \quad \textbf{(85)}$$

Substituting the values of K_1^{DSS} and K_4^{DSS} in Eqns. 6 (a- d) , we have:

$$\frac{P^4A^{2(DSS)}}{64E^2I^{4(DSS)}} [N_1^{CS}N_3^{CS} - N_5^{CS}N_7^{CS}] - \frac{P^3A^{(DSS)}}{16E\,I^{2(DSS)}} [N_1^{CS}N_4^{CS} + N_2^{CS}N_3^{CS} - N_5^{CS}N_8^{CS} - N_6^{CS}N_7^{CS}] + \frac{P^2}{4}[N_2^{CS}N_4^{CS} - N_6^{CS}N_8^{CS}] \quad \textbf{(86)}$$

Eqn.(86) can be further simplified to:

$$P^4\theta_1^{DSS-CS} - \tfrac{1}{2}P^3\theta_2^{DSS-CS} + P^2\theta_3^{DSS-CS} = 0 \quad \textbf{(87)}$$

$$\Longrightarrow P^2 (P^2\theta_1^{DSS-CS} - \tfrac{1}{2}P\theta_2^{DSS-CS} + \theta_3^{DSS-CS}) = 0 \quad \textbf{(88)}$$

Since $P^2 \neq 0$, then, $P^2\theta_1^{DSS-CS} - \tfrac{1}{2}P\theta_2^{DSS-CS} + \theta_3^{DSS-CS} = 0$ or $2P^2\theta_1^{DSS-CS} - P\theta_2^{DSS-CS} + 2\theta_3^{DSS-CS} = 0$ **89(a-b)**

Where

$$\theta_1^{DSS-CS} = C^{K^{2(DSS)}}[N_1^{CS}N_3^{CS} - N_5^{CS}N_7^{CS}] \quad \textbf{(90)}$$

$$\theta_2^{DSS-CS} = C^{K^{(DSS)}}[N_1^{CS}N_4^{CS} + N_2^{CS}N_3^{CS} - N_5^{CS}N_8^{CS} - N_6^{CS}N_7^{CS}] \quad \textbf{(91)}$$

$$\theta_3^{DSS-CS} = \frac{1}{4}[N_2^{CC}N_4^{CC} - N_6^{CC}N_8^{CC}] \quad (92)$$

$C^{K(DSS)}$ = Kings constant for DSS cross section defined in Eqn.(25)

P or P_{DSS}^{F-CS} is the critical buckling load for Flexural –DSS-CS-TWC stability analysis

3.4. NUMERICAL STUDY FOR FLEXURAL [F] - DSS THIN-WALLED COLUMN UNDER DIFFERENT BOUNDARY CONDITION.

A numerical study was carried out on DSS- Thin- Walled Steel Box Column with the following parameters.

E = 210 X 10^3 , G = 81 X 10^3 , L = 4.5m, t varies from 0.0005m , 0.00075m, 0.001m, 0.0025m, 0.005m, 0.0075m, 0.01m, 0.0125m, 0.015m, 0175m to 0.02m, a = 0.08m, where a = column cross section dimension in metres. The main aim is to compare the present result with the one obtained by Ezeh and Osadebe (2010) using Vlasov method. Note that A^{DSS} and I^{DSS} have been defined in Eqns. (7) and (8) respectively.

Using a well-written coordinated algorithm, the required results for the Flexural [F]- DSS buckling analysis for [S-S], [C-C] and [C-S] boundary conditions are shown in Table 1. Table 1 shows the comparison of the Flexural buckling loads for [S-S], [C-C] and [C-S]- DSS thin walled box column between the present study(RRM) and that of Ezeh and Osadede (2010) using Vlasov method.

Table 1: Comparison Between Rayleigh-Ritz Flexural [F] Buckling Values And Vlasov Flexural [F] Critical Buckling Values For DSS Thin Walled Box Column Under Different Boundary Condition.

S/N	P^F / t(m)	Pinned-Pinned [SS] (MN)		Fixed-Pinned[C-S] (MN)		Fixed-Fixed [C-C] (MN)	
		RRM PRESENT STUDY **x E-6**	**VLASOV METHOD** EZEH & OSADEDE (2010)	**RRM** PRESENT STUDY **x E-6**	**VLASOV METHOD** EZEH & OSADEDE (2010)	**RRM** PRESENT STUDY **x E-6**	**VLASOV METHOD** EZEH & OSADEDE (2010)
1	0.02	3.915	23.673	2.026	47.686	8213.0	90.714
2	0.0175	3.709	20.713	1.743	41.725	7137.8	79.374
3	0.015	3.181	17.754	1.500	35.764	6116.2	68.035
4	0.0125	2.662	14.795	1.240	29.803	5070.1	56.696
5	0.01	2.200	11.836	1.026	23.843	4222.7	45.357
6	0.0075	1.585	8.877	0.740	17.882	3025.5	34.018
7	0.005	1.098	5.918	0.500	11.921	1986.4	22.678
8	0.0025	0.525	2.959	0.250	5.951	1015.5	11.339
9	0.001	0.212	1.184	0.100	2.384	407.5	4.536
10	0.00075	0.034	0.888	0.075	1.788	304.8	3.402
11	0.0005	0.023	0.592	0.050	1.192	203.6	2.263

4. CONCLUSIONS

So far in this research study, Rayleigh –Ritz Method (RRM) as a classical energy method for resolving structural stability problems has been presented. The RRM- based formulated DSS TPEF equations derived in the previous works by Nwachukwu and others (2017) and Nwachukwu and others (2021a) were subjected to stability analysis where stability matrices were formed with respect to different boundary conditions. Using a well-coordinated algorithm developed in the course of this work to enable handy solutions to flexural buckling equations, the stability matrices were solved for Flexural [F] buckling for the DSS cross sections under Pinned-Pinned [S-S], Fixed-Pinned[C-S] , Fixed-Fixed [C-C] boundary conditions. The results of the flexural stability analysis (critical buckling loads) of the DSS cross section are as depicted in Table 1. The results are well suited as design data for DSS TWC designs if all other things remain constant. For Table 1, the RRM (present study) critical buckling values are compared with the works of Ezeh and Osadebe (2010) with respect to Pinned-Pinned [S-S], Fixed-Pinned[C-S] , Fixed-Fixed [C-C] boundary conditions for the Flexural buckling load. It can be envisaged from Table 1 that RRM buckling value increases as the thickness of the TWC increases. That means that there is a direct relationship between the RRM buckling values and thickness of the TWC. Although, the comparison between the methods are not too close, but the ability of the RRM buckling values to make TWC structures stable are guaranteed as any TWS designed with values little above the critical buckling values will surely satisfy the engineering safety criteria. Thus, TWC designers should be encouraged to use estimated values that are a little bit greater than P^{crit} for the selection of suitable profile as well as structural design of the TWC to ensure proper safety of the structure.

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