ISSN 2349-7815

International Journal of Recent Research in Electrical and Electronics Engineering (IJRREEE)
Vol. 10, Issue 3, pp: (5-9), Month: July - September 2023, Available at: www.paperpublications.org

A New Definition of Fractional Calculus

Chii-Huei Yu

School of Mathematics and Statistics, Zhaoqing University, Guangdong, China

DOI: https://doi.org/10.5281/zenodo.8314984

Published Date: 04-September-2023

Abstract: **In this paper, based on a new multiplication of fractional analytic functions, we define a new fractional calculus. Moreover, we provide some properties of this new fractional calculus. In fact, our results are generalizations of classical calculus results.**

Keywords: **New multiplication, fractional analytic functions, new fractional calculus.**

I. INTRODUCTION

Fractional calculus belongs to the field of mathematical analysis, involving the research and applications of arbitrary order integrals and derivatives. Fractional calculus originated from a problem put forward by L'Hospital and Leibniz in 1695. Therefore, the history of fractional calculus was formed more than 300 years ago, and fractional calculus and classical calculus have almost the same long history. Since then, fractional calculus has attracted the attention of many contemporary great mathematicians, such as N. H. Abel, M. Caputo, L. Euler, J. Fourier, A. K. Grunwald, J. Hadamard, G. H. Hardy, O. Heaviside, H. J. Holmgren, P. S. Laplace, G. W. Leibniz, A. V. Letnikov, J. Liouville, B. Riemann, M. Riesz, and H. Weyl. With the efforts of researchers, the theory of fractional calculus and its applications have developed rapidly. On the other hand, fractional calculus has wide applications in continuum mechanics, quantum mechanics, electrical engineering, viscoelasticity, control theory, dynamics, economics, and other fields [1-16].

However, the rule of fractional derivative is not unique, many scholars have given the definitions of fractional derivatives. The common definition is Riemann-Liouville (R-L) fractional derivatives. Other useful definitions include Caputo fractional derivatives, Grunwald-Letnikov (G-L) fractional derivatives, and Jumarie type of R-L fractional derivatives to avoid non-zero fractional derivative of constant function [17-21].

In this paper, based on a new multiplication of fractional analytic functions, we define a new fractional calculus. In fact, the new fractional calculus is a generalization of classical calculus.

II. PRELIMINARIES

Definition 2.1: If x, x_0, and a_n are real numbers for all n, $x_0 \in (a,b)$, and $0<\alpha\leq 1$. If the function $f_\alpha: [a,b] \to R$ can be expressed as an α-fractional power series, that is, $f_\alpha\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right) = \sum_{n=0}^{\infty} \frac{a_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha - \frac{1}{\Gamma(\alpha+1)}x_0^\alpha\right)^{\otimes_\alpha n}$ on some open interval containing x_0, then we say that $f_\alpha\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)$ is α-fractional analytic at x_0. Furthermore, if $f_\alpha: [a,b] \to R$ is continuous on closed interval $[a,b]$ and it is α-fractional analytic at every point in open interval (a,b), then f_α is called an α-fractional analytic function on $[a,b]$.

In the following, we introduce a new multiplication of fractional analytic functions.

Definition 2.2: If $0<\alpha\leq 1$. Assume that $f_\alpha\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)$ and $g_\alpha\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)$ are two α-fractional power series at $x=x_0$,

$$f_\alpha\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)=\sum_{n=0}^{\infty}\frac{a_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha-\frac{1}{\Gamma(\alpha+1)}x_0^\alpha\right)^{\otimes_\alpha n},\tag{1}$$

$$g_\alpha\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)=\sum_{n=0}^{\infty}\frac{b_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha-\frac{1}{\Gamma(\alpha+1)}x_0^\alpha\right)^{\otimes_\alpha n}.\tag{2}$$

Then

$$\begin{aligned}&f_\alpha\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)\otimes_\alpha g_\alpha\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)\\&=\sum_{n=0}^{\infty}\frac{a_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha-\frac{1}{\Gamma(\alpha+1)}x_0^\alpha\right)^{\otimes_\alpha n}\otimes_\alpha\sum_{n=0}^{\infty}\frac{b_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha-\frac{1}{\Gamma(\alpha+1)}x_0^\alpha\right)^{\otimes_\alpha n}\\&=\sum_{n=0}^{\infty}\frac{1}{n!}\left(\sum_{m=0}^{n}\binom{n}{m}a_{n-m}b_m\right)\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha-\frac{1}{\Gamma(\alpha+1)}x_0^\alpha\right)^{\otimes_\alpha n}.\end{aligned}\tag{3}$$

Definition 2.3 ([22])**:** If $0<\alpha\le 1$, and x is a real number. The α-fractional exponential function is defined by

$$E_\alpha\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)=\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha n}.\tag{4}$$

On the other hand, the α-fractional cosine and sine function are defined as follows:

$$cos_\alpha\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha(2n)},\tag{5}$$

and

$$sin_\alpha\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha(2n+1)}.\tag{6}$$

In the following, we introduce a new fractional derivative.

Definition 2.4: Let $0<\alpha\le 1$, x be a real number and $\Gamma(\ \ )$ be the gamma function. The new α-fractional derivative is defined by

$$\frac{d^\alpha}{d\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]}\left[f\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)\right]=\lim_{h\to 0}\left[f\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha+\frac{1}{\Gamma(\alpha+1)}h^\alpha\right)-f\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)\right]\otimes_\alpha\left(\frac{1}{\Gamma(\alpha+1)}h^\alpha\right)^{\otimes_\alpha(-1)}.\tag{7}$$

Moreover, we define

$$\left(\frac{d^\alpha}{d\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]}\right)^p\left[f\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)\right]=\left(\frac{d^\alpha}{d\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]}\right)\left(\frac{d^\alpha}{d\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]}\right)\cdots\left(\frac{d^\alpha}{d\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]}\right)\left[f\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)\right]\tag{8}$$

is the p-th order α-fractional derivative of $f(x)$, for any positive integer p.

In the following, a new fractional integral is introduced.

Definition 2.5: Let $0<\alpha<1$, and $f:[a,b]\to R$. If

$$\lim_{\|\Delta\|\to 0}\sum_{k=1}^{n}f\left(\frac{1}{\Gamma(\alpha+1)}\xi_k^\alpha\right)\otimes_\alpha\left(\frac{1}{\Gamma(\alpha+1)}x_k^\alpha-\frac{1}{\Gamma(\alpha+1)}x_{k-1}^\alpha\right)$$

exists, where the partitions of the interval $[a,b]$ are denoted by $[x_{k-1},x_k]$, $k=1,\cdots,n$, $x_0=a$, $x_n=b$, $\xi_k\in[x_{k-1},x_k]$, $\Delta x_k=x_k-x_{k-1}$, and $\|\Delta\|=\max_{k=1,\cdots,n}\{\Delta x_k\}$. Then we say f is a α-fractional integrable function on $[a,b]$. And it is denoted by

$$\lim_{\|\Delta\|\to 0}\sum_{k=1}^{n}f\left(\frac{1}{\Gamma(\alpha+1)}\xi_k^\alpha\right)\otimes_\alpha\left(\frac{1}{\Gamma(\alpha+1)}x_k^\alpha-\frac{1}{\Gamma(\alpha+1)}x_{k-1}^\alpha\right)=\int_a^b f\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)d\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right),\tag{9}$$

which is called the α-fractional integral of f on $[a,b]$.

ISSN 2349-7815
International Journal of Recent Research in Electrical and Electronics Engineering (IJRREEE)
Vol. 10, Issue 3, pp: (5-9), Month: July - September 2023, Available at: www.paperpublications.org

III. RESULTS

In this section, we provide some properties of the new fractional calculus.

Theorem 3.1: *If* α, C *are real numbers,* $0 < \alpha \leq 1$*, and* n *is a positive integer, then*

$$\frac{d^{\alpha}}{d\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]}[C] = 0, \tag{10}$$

$$\frac{d^{\alpha}}{d\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]}\left[\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_{\alpha} n}\right] = n\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_{\alpha}(n-1)}, \tag{11}$$

$$\frac{d^{\alpha}}{d\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]}\left[E_{\alpha}\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)\right] = E_{\alpha}\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right), \tag{12}$$

$$\frac{d^{\alpha}}{d\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]}\left[sin_{\alpha}\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)\right] = cos_{\alpha}\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right), \tag{13}$$

$$\frac{d^{\alpha}}{d\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]}\left[cos_{\alpha}\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)\right] = -sin_{\alpha}\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right). \tag{14}$$

Proof $\frac{d^{\alpha}}{d\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]}[C] = \lim_{h\to 0}[C-C]\otimes_{\alpha}\left(\frac{1}{\Gamma(\alpha+1)}h^{\alpha}\right)^{\otimes_{\alpha}(-1)} = \lim_{h\to 0}[0]\otimes_{\alpha}\left(\frac{1}{\Gamma(\alpha+1)}h^{\alpha}\right)^{\otimes_{\alpha}(-1)} = 0.$

Next,

$$\frac{d^{\alpha}}{d\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]}\left[\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_{\alpha} n}\right]$$

$$= \lim_{h\to 0}\left[\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha} + \frac{1}{\Gamma(\alpha+1)}h^{\alpha}\right)^{\otimes_{\alpha} n} - \left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_{\alpha} n}\right]\otimes_{\alpha}\left(\frac{1}{\Gamma(\alpha+1)}h^{\alpha}\right)^{\otimes_{\alpha}(-1)}$$

$$= \lim_{h\to 0}\left[\begin{array}{c}\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_{\alpha} n} + \binom{n}{1}\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_{\alpha}(n-1)}\otimes_{\alpha}\left(\frac{1}{\Gamma(\alpha+1)}h^{\alpha}\right) + \\ \binom{n}{2}\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_{\alpha}(n-2)}\otimes_{\alpha}\left(\frac{1}{\Gamma(\alpha+1)}h^{\alpha}\right)^{\otimes_{\alpha} 2} \\ + \cdots + \left(\frac{1}{\Gamma(\alpha+1)}h^{\alpha}\right)^{\otimes_{\alpha} n} - \left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_{\alpha} n}\end{array}\right]\otimes_{\alpha}\left(\frac{1}{\Gamma(\alpha+1)}h^{\alpha}\right)^{\otimes_{\alpha}(-1)}$$

$$= \lim_{h\to 0}\left[\binom{n}{1}\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_{\alpha}(n-1)} + \binom{n}{2}\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_{\alpha}(n-2)}\otimes_{\alpha}\frac{1}{\Gamma(\alpha+1)}h^{\alpha} + \cdots + \left(\frac{1}{\Gamma(\alpha+1)}h^{\alpha}\right)^{\otimes_{\alpha}(n-1)}\right]$$

$$= n\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_{\alpha}(n-1)}.$$

Moreover,

$$\frac{d^{\alpha}}{d\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]}\left[E_{\alpha}\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)\right]$$

$$= \frac{d^{\alpha}}{d\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]}\left[\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_{\alpha} n}.\right]$$

$$= \sum_{n=0}^{\infty}\frac{1}{n!}\cdot\frac{d^{\alpha}}{d\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]}\left[\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_{\alpha} n}\right]$$

$$= \sum_{n=1}^{\infty}\frac{1}{n!}\cdot n\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_{\alpha}(n-1)}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{1}{\Gamma(\alpha+1)} x^{\alpha}\right)^{\otimes_\alpha n}$$

$$= E_{\alpha}\left(\frac{1}{\Gamma(\alpha+1)} x^{\alpha}\right).$$

And

$$\frac{d^{\alpha}}{d\left[\frac{1}{\Gamma(\alpha+1)} x^{\alpha}\right]}\left[\sin_{\alpha}\left(\frac{1}{\Gamma(\alpha+1)} x^{\alpha}\right)\right]$$

$$= \frac{d^{\alpha}}{d\left[\frac{1}{\Gamma(\alpha+1)} x^{\alpha}\right]}\left[\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\left(\frac{1}{\Gamma(\alpha+1)} x^{\alpha}\right)^{\otimes_\alpha (2n+1)}\right]$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \cdot \frac{d^{\alpha}}{d\left[\frac{1}{\Gamma(\alpha+1)} x^{\alpha}\right]}\left[\left(\frac{1}{\Gamma(\alpha+1)} x^{\alpha}\right)^{\otimes_\alpha (2n+1)}\right]$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \cdot (2n+1) \cdot \left(\frac{1}{\Gamma(\alpha+1)} x^{\alpha}\right)^{\otimes_\alpha (2n)}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\left(\frac{1}{\Gamma(\alpha+1)} x^{\alpha}\right)^{\otimes_\alpha (2n)}$$

$$= \cos_{\alpha}\left(\frac{1}{\Gamma(\alpha+1)} x^{\alpha}\right).$$

Finally,

$$\frac{d^{\alpha}}{d\left[\frac{1}{\Gamma(\alpha+1)} x^{\alpha}\right]}\left[\cos_{\alpha}\left(\frac{1}{\Gamma(\alpha+1)} x^{\alpha}\right)\right]$$

$$= \frac{d^{\alpha}}{d\left[\frac{1}{\Gamma(\alpha+1)} x^{\alpha}\right]}\left[\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\left(\frac{1}{\Gamma(\alpha+1)} x^{\alpha}\right)^{\otimes_\alpha (2n)}\right]$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \cdot \frac{d^{\alpha}}{d\left[\frac{1}{\Gamma(\alpha+1)} x^{\alpha}\right]}\left[\left(\frac{1}{\Gamma(\alpha+1)} x^{\alpha}\right)^{\otimes_\alpha (2n)}\right]$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!} \cdot (2n) \cdot \left(\frac{1}{\Gamma(\alpha+1)} x^{\alpha}\right)^{\otimes_\alpha (2n-1)}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)!} \cdot \left(\frac{1}{\Gamma(\alpha+1)} x^{\alpha}\right)^{\otimes_\alpha (2n-1)}$$

$$= -\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\left(\frac{1}{\Gamma(\alpha+1)} x^{\alpha}\right)^{\otimes_\alpha (2n+1)}$$

$$= -\sin_{\alpha}\left(\frac{1}{\Gamma(\alpha+1)} x^{\alpha}\right).$$

Q.e.d.

IV. CONCLUSION

In this paper, based on a new multiplication of fractional analytic functions, we define a new fractional calculus. On the other hand, we give some properties of this new fractional calculus. Moreover, our results are generalizations of traditional calculus results. In the future, we will continue to use this new fractional calculus to solve problems in applied mathematics and fractional differential equations.

REFERENCES

[1] Carpinteri, F. Mainardi, Fractals and Fractional Calculus in Continuum Mechanics, Springer, New York, 1997.

[2] J. Sabatier, O. P. Agrawal, J.A. Tenreiro Machado, Advances in Fractional Calculus. Theoretical Developments and Applications in Physics and Engineering, Springer, Dordrecht, 2007.

[3] V. E. Tarasov, Review of Some Promising Fractional Physical Models, International Journal of Modern Physics. Vol. 27, No. 9, 2013.

[4] V. V. Uchaikin, Fractional Derivatives for Physicists and Engineers, Vol. 1, Background and Theory, Vol. 2, Application. Springer, 2013.

[5] J. T. Machado, Fractional Calculus: Application in Modeling and Control, Springer New York, 2013.

[6] R. L. Magin, Fractional calculus in bioengineering, 13th International Carpathian Control Conference, 2012.

[7] R. Hilfer (ed.), Applications of Fractional Calculus in Physics, WSPC, Singapore, 2000.

[8] Mohd. Farman Ali, Manoj Sharma, Renu Jain, An application of fractional calculus in electrical engineering, Advanced Engineering Technology and Application, vol. 5, no. 2, pp. 41-45, 2016.

[9] V. E. Tarasov, Mathematical economics: application of fractional calculus, Mathematics, Vol. 8, No. 5, 660, 2020.

[10] M. F. Silva, J. A. T. Machado, A. M. Lopes, Fractional order control of a hexapod robot, Nonlinear Dynamics, vol. 38, pp. 417-433, 2004.

[11] N. Heymans, Dynamic measurements in long-memory materials: fractional calculus evaluation of approach to steady state, Journal of Vibration and Control, vol. 14, no. 9, pp. 1587-1596, 2008.

[12] R. C. Koeller, Applications of fractional calculus to the theory of viscoelasticity, Journal of Applied Mechanics, vol. 51, no. 2, 299, 1984.

[13] T. Sandev, R. Metzler, & Ž. Tomovski, Fractional diffusion equation with a generalized Riemann–Liouville time fractional derivative, Journal of Physics A: Mathematical and Theoretical, vol. 44, no. 25, 255203, 2011.

[14] J. P. Yan, C. P. Li, On chaos synchronization of fractional differential equations, Chaos, Solitons & Fractals, vol. 32, pp. 725-735, 2007.

[15] -H. Yu, A study on fractional RLC circuit, International Research Journal of Engineering and Technology, vol. 7, no. 8, pp. 3422-3425, 2020.

[16] -H. Yu, A new insight into fractional logistic equation, International Journal of Engineering Research and Reviews, vol. 9, no. 2, pp.13-17, 2021.

[17] K. S. Miller, B. Ross, An Introduction to the Fractional Calculus and Fractional Differential Equations; John Willy and Sons, Inc.: New York, NY, USA, 1993.

[18] K. B. Oldham, J. Spanier, The Fractional Calculus; Academic Press: New York, NY, USA, 1974.

[19] I. Podlubny, Fractional Differential Equations; Academic Press: New York, NY, USA, 1999.

[20] S. Das, Functional Fractional Calculus, 2nd Edition, Springer-Verlag, 2011.

[21] K. Diethelm, The Analysis of Fractional Differential Equations, Springer-Verlag, 2010.

[22] U. Ghosh, S. Sengupta, S. Sarkar, and S. Das, Analytic solution of linear fractional differential equation with Jumarie derivative in term of Mittag-Leffler function, American Journal of Mathematical Analysis, vol. 3, no. 2, pp.32-38, 2015.